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ABSTRACT

This fourth unit in the SMSG junior high mathematics series is the teacher's commentary for Unit 2. A time allotment for each of the chapters in Unit 2 is suggested. Then, for each of the chapters in Unit 2, the objectives for that chapter are specified, the mathematics is discussed, some teaching suggestions are provided, the answers to exercises are listed, and sample test questions for that chapter are suggested. (DT)

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School Mathematics Study Group

Mathematics for Junior High School, Volume 1

Unit 4

Mathematics for Junior High School, Volume 1

Teacher's Commentary, Part II

REVISED EDITION

Prepared under the supervision of the
Panel on Seventh and Eighth Grades
of the School Mathematics Study Group:

R. D. Anderson	Louisiana State University
J. A. Brown	University of Delaware
Lenore John	University of Chicago
B. W. Jones	University of Colorado
P. S. Jones	University of Michigan
J. R. Mayor	American Association for the Advancement of Science
P. C. Rosenbloom	University of Minnesota
Veryl Schult	Supervisor of Mathematics, Washington, D.C.

Stanford, California
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The following is a list of all those who have participated in the preparation of this volume.

R.D. Anderson, Louisiana State University

B.H. Arnold, Oregon State College

J.A. Brown, University of Delaware

Kenneth E. Brown, U.S. Office of Education

Mildred B. Cole, K.D. Waldo Junior High School, Aurora, Illinois

B.H. Colvin, Boeing Scientific Research Laboratories

J.A. Cooley, University of Tennessee

Richard Dean, California Institute of Technology

H.M. Gehman, University of Buffalo

L. Roland Genise, Brentwood Junior High School, Brentwood, New York

E. Glenadine Gibb, Iowa State Teachers College

Richard Good, University of Maryland

Alice Hach, Racine Public Schools, Racine, Wisconsin

S.B. Jackson, University of Maryland

Lenore John, University High School, University of Chicago

B.W. Jones, University of Colorado

P.S. Jones, University of Michigan

Houston Karnes, Louisiana State University

Mildred Keiffer, Cincinnati Public Schools, Cincinnati, Ohio

Nick Lovdjieff, Anthony Junior High School, Minneapolis, Minnesota

J.R. Mayor, American Association for the Advancement of Science

Sheldon Meyers, Educational Testing Service

Muriel Mills, Hill Junior High School, Denver, Colorado

P.C. Rosenbloom, University of Minnesota

Elizabeth Roudebush, Seattle Public Schools, Seattle, Washington

Veryl Schult, Washington Public Schools, Washington, D.C.

George Schaefer, Alexis I. DuPont High School, Wilmington, Delaware

Allen Shields, University of Michigan

Rothwell Stephens, Knox College

John Wagner, School Mathematics Study Group, New Haven, Connecticut

Ray Walch, Westport Public Schools, Westport, Connecticut

G.C. Webber, University of Delaware

A.B. Willcox, Amherst College

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NOTE TO TEACHERS

Based on the teaching experience of over 100 junior high school teachers in all parts of the country and the estimates of authors of the revisions (including junior high school teachers), it is recommended that teaching time for Part 2 be as follows:

Chapter	Approximate number of days
9	18
10	12
11	13
12	11
13	10
14	5
Total	69

Throughout the text, problems, topics, and sections which were designed for the better students are indicated by an asterisk (*). Items starred in this manner should be used or omitted as a means of adjusting the approximate time schedule.

Chapter 9

RATIOS, PERCENTS, AND DECIMALS

The important concept of this chapter is ratio. When a physical or mathematical law can be written as a proportion, then this law can be used to deduce new information from old. Thus in our first example, shadow length is proportional to height. Suppose we know the ratio of the measure of the shadow length to the measure of a corresponding height. If we are given another shadow length we can find the corresponding height, or if we are given another height we can find the corresponding shadow length.

The ideas behind the words ratio, percent, and decimal enable us to express old and familiar properties of rational numbers in a convenient way. A ratio may be expressed by a rational number. A rational number may be expressed as a percent or in decimal notation. Naturally, we have to spend a great deal of time helping students in developing sufficient skill with these new notations for the students to become really competent in using the notations.

Certainly there are important and interesting results to be established. No one will call the theorem that only rational numbers have repeating decimal expansions a trivial result. The correspondence of a point on the number line with a decimal expansion is an exciting and a far-reaching result. On the other hand, decimals are simply notations and add only a little to our understanding of rational numbers as a mathematical system. Proportion, however, is a new concept, essentially having its origin in physical examples.

The examples of this chapter will provide the students with situations where it is a natural thing to introduce the notations of ratio, percent, and decimals as handy and efficient short cuts of expression. It is our hope that if a purpose is seen for their introduction, the teaching of the technique of their use will prove easier.

Many teachers who have used the SMSG texts in grades 7 and 8 believe that the attention given to decimals and percents and their applications is sufficient. Others are of the opinion that more time for the study of these topics is needed if their students are to achieve the degree of mastery of them which has become traditional. Special effort has been made in the revision of this chapter to bring about greater efficiency in the use of ratio, percents, and decimals. Additional experience with ratio, percents, and decimals has been provided in exercises in the remainder of this text and the text prepared for use in grade 8. The principal difference in treatment in the SMSG and the traditional texts is that in SMSG materials these topics are more closely related to the properties of number systems from a mathematical point of view, and less time is given to a discussion of the social situations in which the various applications arise.

Based on the teaching experience in the 12 centers for junior high school mathematics, the writing committee recommends that 17 or 18 teaching days be used in teaching this chapter.

9-1. Ratios and Proportion

This section should not take more than two class days. Class discussions may well include some simple calculations that can be done in a few minutes. Most teachers will divide the rather long set of exercises into two assignments.

The idea of ratio was introduced in Chapter 6. The purpose of this section is to provide additional work with ratios and to give a meaningful introduction to proportion. Many applications are included. It is recommended that the teacher emphasize that when a mathematical or physical law can be written as a proportion, then this law can be used to obtain new information and that the use of proportion involves only quite easy mathematical ideas.

[pages 347-350]

The authors do not recommend that a particular method be used in solving problems involving two equal ratios (proportions). Some answers can be obtained by mental computation. This should not be discouraged. In finding n in the equation (proportion)

$$\frac{n}{18} = \frac{6}{5}$$

some students may wish to multiply $\frac{n}{18}$ and $\frac{6}{5}$ by the number one in some form so that the two fractions $\frac{n}{18}$ and $\frac{6}{5}$ have the same denominator.

In the last step, remind the students of the definition of a rational number in Chapter 6.

Other students may prefer to use the property that if $\frac{a}{b} = \frac{c}{d}$, $b \neq 0$ and $d \neq 0$, then $ad = bc$, and only then. The teacher will of course recognize this property as the familiar "product of the means equals the product of the extremes." We have carefully avoided the familiar expression since it is a special device for a special situation and in this form can not be so easily related to the mathematical structures which the student should constantly be making a part of his way of thinking.

Attention of the teacher is called to the significance of only then in the statement of this property. The property as stated above and in Exercise *18 illustrates what is known as a necessary and sufficient condition, or an if and only if condition in mathematics. There are two separate properties to be proved. The first is proved in the text and the proof of the second is left as a problem. See Exercise *18. In some classes the important distinction between necessary and sufficient, or if and only if, may be meaningful only to the better students.

The authors strongly recommend that the notation a:b::c:d not be used.

Problems *14, *15, *16, *17 are sometimes called problems involving multiple ratios. It is recommended that these problems not be considered a separate type of problem situation. The teacher, however, may wish to give special attention in class to a problem like these.

Answers to Exercises 9-1

1. Depends on height of individuals. Length of shadow is $\frac{2}{3}$ × height.

2.

3	8	$\frac{3}{8}$
36	96	$\frac{3}{8}$
$7\frac{1}{2}$	20	$\frac{3}{8}$
54	144	$\frac{3}{8}$
$11\frac{1}{4}$	30	$\frac{3}{8}$

3. (a) Ratio of number of girls to total is $\frac{2}{5}$.
   (b) Ratio of number of boys to total is $\frac{3}{5}$.
   (c) Ratio of number of girls to number of boys is $\frac{2}{3}$.

4. 15 boys

5. The following ratios are equal:
   (a), (c), (d), (e).

6. (a) $x = 15$
   (b) $x = 42$
   (c) $x = 15$
   (d) $x = 9$
   (e) $x = 56$

7. 10 inches long

8. 112.5 ft. tall

9. $202.50

10. In the ratio 3:1 we have:

3 cups butter
2 cups sugar
6 eggs
$4\frac{1}{2}$ cups flour
3 teaspoons vanilla

It will now make 90 cookies.
To make 45 cookies you would rewrite the recipe in the ratio $1\frac{1}{2}$:1

$1\frac{1}{2}$ cups butter
1 cup sugar
3 eggs
$2\frac{1}{4}$ cups flour
$1\frac{1}{2}$ teaspoon vanilla

11. (a) $1.65

(b) 45 cents (45 cents per dozen)

(c) $119

(d) 316.8 feet or 317 to the nearest foot.

12. 44 feet per second

13.

(a)	12	14	$\frac{12}{14}$	$\frac{6}{7}$
(b)	18	21	$\frac{18}{21}$	$\frac{6}{7}$
(c)	30	35	$\frac{30}{35}$	$\frac{6}{7}$
(d)	$85\frac{5}{7}$	100	$\frac{85\frac{5}{7}}{100}$	$\frac{6}{7}$
(e)	100	$116\frac{2}{3}$	$\frac{100}{116\frac{2}{3}}$	$\frac{6}{7}$

*14. This is a starred section, to be used with some students if time permits. To make 24 pounds of a mixture of peanuts, cashews, and pecans in a ratio of 5 to 2 to 1 the grocer would use 15 pounds of peanuts, 6 pounds of cashews, and 3 pounds of pecans. The answers to the questions in the discussion are:

1. 8 pounds of nuts.
2. 5 to 8 is the ratio of peanuts to total.
3. 15 pounds of peanuts.
4. Cashews 2 to 8 is ratio to total
   6 pounds of cashews.
5. Pecans 1 to 8 is ratio of pecans to total
   3 pounds of pecans.

*15. 35 pounds of peanuts, 14 lbs. of cashews, 7 lbs. of pecans.

*16. 50 pounds peanuts, 30 pounds cashews, 20 pounds pecans.

*17. 24 inches.

9-2. Percent

The concepts of percent are introduced briefly in Section 2. In Sections 7 and 8 of this chapter, percent will be treated more fully. In both of the sections the meaning of percent is based on the idea that "a"% means $\frac{a}{100} = a \times \frac{1}{100}$. All three "cases" of percent are introduced informally in Section 2 with numbers that are easily handled. However, you will notice that the three "cases" of percent are not referred to in this textbook. Notice that the solutions of all problems are set up in the form $\frac{a}{b} = \frac{x}{100}$. The method of solving for x should be the method that the pupil understands. Here we have used Property 1 from Section 1 and the property that if $ax = b$ then $x = \frac{b}{a}$. Pupils should be encouraged to use any method they understand.

[pages 353-356]

In the example of the class of 11 girls and 14 boys, the fraction $\frac{44}{100}$ indicates percent easily because the denominator is 100. The sum of $\frac{11}{25}$ and $\frac{14}{25}$ is one. The sum of the ratios $\frac{44}{100}$ and $\frac{56}{100}$ is one, and the sum of 44% and 56% is 100% or one.

Exercises 9-2a emphasize writing percents first as fractions with 100 as the denominator, and then with the % symbol. Another point of emphasis is that 100% stands for one or the whole of a quantity.

Answers to Exercises 9-2a

1. (a) $\frac{1}{10}$
   (b) 10%
   (c) 20
   (d) $\frac{1}{5}$
   (e) 35
   (f) $\frac{7}{20}$
   (g) $\frac{3}{10}$
   (h) 30%
   (i) $\frac{1}{20}$
   (j) 5%
2. One
3. 100%
4. (a) 50%
   (b) 25%
   (c) 75%
   (d) 20%
   (e) 40%
   (f) 60%
   (g) 150%
   (h) 100%
   (i) 100%
   (j) 140%
   (k) 125%
   (ℓ) 250%
5. (a) 60%
   (b) 30%
   (c) 20%
   (d) 80%
   (e) 20%
   (f) 10%

6. (a) $\frac{12}{50}$ or $\frac{6}{25}$ or $\frac{24}{100}$, 24%; $\frac{10}{50}$ or $\frac{1}{\text{[illegible]}}$ or $\frac{20}{100}$, 20%; $\frac{15}{50}$ or $\frac{3}{10}$ or $\frac{30}{100}$, 30%; $\frac{13}{50}$ or $\frac{26}{100}$, 26%.

(b) 1 (c) 100%

(d) The ratios total 1, and the percents total 100%.

7. (a) Mortgage 20%; taxes 5%; car 9%; food 30%; clothing 12%; operating 8%; health and recreation 6%; savings and insurance 10%.

(b) The 8 answers total 100%.

The second lesson in 9-2 emphasizes the use of percent for purposes of comparison, and for giving information in more usable form. Pupils should learn to calculate mentally 1% of a quantity, and also 10%. Frequent oral practice should include such calculations.

Answers to Exercises 9-2b

1. (a) 60% (b) 40%
2. (a) 50% or $\frac{50}{100}$ (b) decreased
3. (a) 65% or $\frac{65}{100}$ (b) higher than both
4. (a) 30 (c) 6
   (b) 5% (d) 60
5. 6%
6. (a) 25% (b) 75%
7. (a) $\frac{1}{10}$ (b) 10%
8. 125%
9. $\frac{1}{2}$%, or $\frac{1}{2}$ of 1%.
10. 550 hint $\frac{22}{x} = \frac{4}{100}$

9-3. Decimal Notation

Presumably all students will have had an introduction to decimals in sixth grade. An attempt is made here to extend the Place Value Chart which is commonly used in earlier grades to include notation used in the expanded form. Thus we have the place name in words, decimal notation, and as a power of ten.

The purpose of this section is to review base and place value, reading and writing rational numbers in decimal form, and to extend the notation for the expanded form.

The teacher could illustrate changing a rational number into other bases.

Example 1. Write $\frac{1}{4}$ as a duodecimal.

There are 12 twelfths in one unit, so in $\frac{1}{4}$ of a unit, there are $\frac{12}{4}$, or 3 twelfths, and none left over.

$\frac{1}{4} = \frac{3}{12} = 0.3_{\text{twelve}}$.

Example 2. Write $\frac{5}{8}$ as a duodecimal.

There are 12 twelfths in one unit and 60 twelfths in 5 units, so in $\frac{5}{8}$ of a unit, there are $\frac{60}{8}$ twelfths or $7\frac{1}{2}$ twelfths. The twelfths digit is "7" and there is $\frac{1}{2}$ of a twelfth left over. There are 12 one-hundred-forty-fourths in one twelfth of a unit, so in $\frac{1}{2}$ of a twelfth, there are $\frac{12}{2}$, or 6 one-hundred-forty-fourths and none left over. The one-hundred-forty-fourths digit is "6".

$\frac{5}{8} = 0.76_{\text{twelve}}$.

Answers to Exercises 9-3

1. (a) 65.87
   (b) 436.19
   (c) 50.24
   (d) 0.483
   (e) 0.0026
   (f) 0.30005
   (g) 300.04

2. (a) $5(10) + 2(1) + 5\left(\frac{1}{10}\right) + 5\left(\frac{1}{10^2}\right)$

(b) $1(1) + 2\left(\frac{1}{10}\right) + 1\left(\frac{1}{10^2}\right) + 3\left(\frac{1}{10^3}\right)$

(c) $4\left(\frac{1}{10}\right)$

(d) $3(1) + 1\left(\frac{1}{10^2}\right)$

(e) $1\left(\frac{1}{10^2}\right) + 2\left(\frac{1}{10^4}\right)$

(f) $1\left(\frac{1}{10}\right) + 1\left(\frac{1}{10^5}\right)$

(g) $3(10) + 3\left(\frac{1}{10^2}\right)$

3. (a) Seven and two hundred thirty-six thousandths.

(b) Four thousandths.

(c) Three hundred sixty and one hundred one thousandths.

(d) One and one hundred one ten-thousandths.

(e) Nine hundred nine and nine thousandths.

(f) Three and forty-four ten-thousandths.

4. (a) 300.52 (d) 60.07

(b) 0.0507 (e) 0.00032

(c) 0.014 (f) 8.019

*5. 0.5

*6. 0.6_{twelve}

*7. $0.T_{\text{twelve}}$

*8. $10.011_{\text{two}} = 1(2) + 1\left(\frac{1}{2^2}\right) + 1\left(\frac{1}{2^3}\right) =$

$= 2 + \frac{1}{4} + \frac{1}{8} = 2 + \frac{2}{8} + \frac{1}{8} =$

$= 2\frac{3}{8} = 2.375.$

9-4. Arithmetic Operations with Decimals

This section develops the arithmetic operations (addition, subtraction, multiplication and division) for terminating decimals. It is assumed that the students have had previous experience with decimals but that they are now at the point where they can look a little more deeply into their meaning and the reasons back of the operations on them. This also serves as a little review of some of the operations on fractions.

Which decimals terminate? First of all, any terminating decimal is a repeating decimal (since 0 is repeated), and represents a rational number. But not all rational numbers have terminating decimals, for example, $\frac{2}{7} = ,285714285714285...$ is not terminating. The terminating decimals are precisely those rational numbers that can be written with their denominators a power of ten. For example, $\frac{1}{5}$ does not have a denominator that is a power of ten. But it can be written in that form:

$$\frac{1}{5} = \frac{2}{10} = 0.2000000000\ldots .$$

So the concept of terminating decimal is a special one that depends on the particular base being used. Thus, if we used the base 7, all fractions that could be written with denominator a power of seven would give rise to terminating decimals.

Thus, the concept of the infinite decimal expansion is the really important concept, while the terminating decimal is a special concept that depends on the particular base being used. However, even with the infinite decimal there is some ambiguity. Thus, $\frac{1}{2}$ has two different representations as a decimal:

$$\frac{1}{2} = 0.500000000 \ldots = 0.4999999999 \ldots$$

The decimals for which this ambiguity arises are precisely the terminating decimals. If a decimal terminates (i.e., has all zeros from some point on), then it can be written with all nines from some point on.

Answers to Exercises 9-4a

1. (a) 1.60 (c) 1.0122
   (b) 1.101 (d) 23.30
2. (a) .08 (c) .375
   (b) .075 (d) 1.0045
3. (a) .415 (b) .163
4. No. They want a total of 50.4 feet of wire.
5. 26.05 pounds of sugar are left.
6. $\frac{7}{16} = 0.4375$, and therefore, 0.45 lb. is greater than 7 ounces.
7. 150.7 kilometers.
8. 75.7 kilometers.

*9. $$\begin{array}{r} 10.01_{two} \\ 1.01_{two} \\ \hline 11.10_{two} \end{array}$$ $11.1_{two} = 1(2^1) + 1 + \frac{1}{2} = 3\frac{1}{2}$ or 3.5

Answers to Exercises 9-4b

1. (a) 0.00081 (c) 144.
   (b) 0.00625 (d) 0.027270
2. (a) 0.0003 (c) 1.4375
   (b) 0.3 (d) 255.
3. 375, 37.5, 3.75, 0.375, 0.0375, 0.00375.
4. 0.0625.
5. 397.21.
6. 80.94 square meters.
7. 358.975 cubic meters.
8. About 2.4 miles

*9. 3.102_{seven} (In the multiplication table, base seven, we have:

$$4_{seven} \times 4_{seven} = 22_{seven}, \text{ and}$$

$$2_{seven} \times 4_{seven} = 11_{seven},$$

9-5. Decimal Expansion

Through many examples pupils may proceed by induction to an affirmative answer to the question: May every rational number be named by a decimal numeral?

This section gives an opportunity for more work in operations with decimals - especially division.

The symbolism for repeating over and over again ... is the same as was introduced in Chapter 1. The bar over the digits that repeat in the same order is new.

Answers to Questions in Class Discussion

1. 1, 4, 2, 8, 5, 7 .
2. Yes.
3. $0.0909\overline{09}$ ...
4. After the first subtraction.
5. No.
6. Yes; by a bar.
8. $0.270\overline{270}$ ...

Answers to Exercises 9-5

1. $0.076923\overline{0769230}$ ...

   (a) After the 5th subtraction.

   (b) No.

   (c) By dots.

   (d) Yes.

   (e) By a bar.

2. (a) $0.3\overline{3}$ ... (d) $0.8750\overline{0}$ ...

   (b) $0.250\overline{0}$ ... (e) $0.1\overline{1}$ ...

   (c) $0.16\overline{6}$ ...

3. (a) $0.09\overline{09}$ ... (d) $0.81\overline{81}$ ...

   (b) $0.18\overline{18}$ ... (e) $1.27\overline{27}$ ...

   (c) $0.27\overline{27}$ ... (f) $2.09\overline{09}$ ...

4. (a) The number $0.18\overline{18}$ ... is twice the number $0.09\overline{09}$ ...

   (b) The number $0.27\overline{27}$ ... is three times the number $0.09\overline{09}$ ...
   The number $0.81\overline{81}$ ... is nine times the number $0.09\overline{09}$ ...
   The number $1.27\overline{27}$ ... is fourteen times the number $0.09\overline{09}$ ...

5. Yes. $5(0.09\overline{09}$ ...) or $0.45\overline{45}$ ...

6. Yes.

7. (a) 0.2, 0.4, 0.8

   (b) 0.125, 0.375, 0.875

   (c) 0.05, 0.15, 0.55

   (d) 0.02, 0.84, 0.94

   (e) 0.001, 0.112, 0.927

9-6. Rounding

It is not intended at this point to consider approximate error but merely approximations to decimal values from an informal point of view. Use is made of rounding in the following sections. The teacher may want to call the attention of the class more strongly to the fact that one can estimate the answer to a complicated problem by means of rounding. Rounding also helps in locating the decimal point.

Answers to Exercises 9-6

1. (a) 0.04 (c) 0.01
   (b) 0.04 (d) 0.02
2. (a) 0.160 (c) 0.000
   (b) 0.001 (d) 0.325
3. (a) 0.375 (b) 0.250 (c) 0.667
4. (a) 0.3 (c) 0.1
   (b) 0.3 (d) 0.0
5. (a) 43.30 sq. rd.
   (b) 43.89 sq. rd.
   0.59 sq. rd. difference

9-7. Percent and Decimals

The purpose of this section is to establish the relationship among the three notations: fractions, decimals and percent. There are two phases of this: to understand how they are related and to acquire the technique for translating from one notation to the other. Of fundamental importance is the property of fractions called Property 1 in Chapter 6 which states that

$$\frac{a}{b} = \frac{ka}{kb}$$

for whole numbers a, b, and k, provided neither b nor k is zero. It is not always true in mathematics that technique and understanding go hand in hand but it is certainly true here. It immediately answers such questions as: Why is 50% equal to $\frac{1}{2}$? Answer: because 50% is equal to $\frac{50}{100}$ and $\frac{50}{100}$ is equal to $\frac{1}{2}$. To get a percent as a decimal or fraction one translates into a fraction whose denominator is 100.

We have here carefully avoided any expressions like $0.12\frac{1}{2}$ because the meaning of $\frac{1}{2}$ here is hard to explain and may be misinterpreted and in any case just as simple, and completely unambiguous, is the form 0.125. For such a fraction as $\frac{1}{3}$ we can use the decimal: 0.333 ... . This point need not be raised in class unless a student slips into the notation $0.12\frac{1}{2}$, for instance. If he does suggest this, he should be asked just what the $\frac{1}{2}$ means. It is really $\frac{1}{2}$ of 0.01. Similarly, if one wrote the decimal $0.123\frac{1}{2}$, the $\frac{1}{2}$ would be $\frac{1}{2}$ of 0.001. This is rather awkward to keep track of and hence it is desirable to avoid this notation.

The teacher may need to give more conversion problems than are given here. Some classes may require less.

Answers to Class Exercise 9-7a

1. (a) 12% (c) 1460%
   (b) 40% (d) 120%
2. (a) 0.53 (c) 18.75
   (b) 1.25 (d) 0.03

Answers to Class Exercise 9-7b

1. (a) $62\frac{1}{2}$% or 62.5% (c) 0.8%
   (b) $18\frac{3}{4}$% or 18.75% (d) $47\frac{1}{2}$% or 47.5%

2. (a) 0.625 (c) 0.1625

   (b) 0.008 (d) 0.1875

   Note that $0.16\frac{1}{4}$ and $0.18\frac{3}{4}$ we avoid since they are misleading.

Answers to Class Exercise 9-7c

1. (a) 66% (c) 111%

   (b) 11% (d) 29%

2. (a) $66\frac{2}{3}\%$ (c) $111\frac{1}{9}\%$

   (b) $11\frac{1}{9}\%$ (d) $28\frac{4}{7}\%$

Answers to Class Exercise 9-7d

1. (a) $\frac{2}{3}$ (c) $\frac{101}{400}$

   (b) $\frac{1}{9}$ (d) $\frac{251}{200}$

   Of course many other answers are possible: For instance,

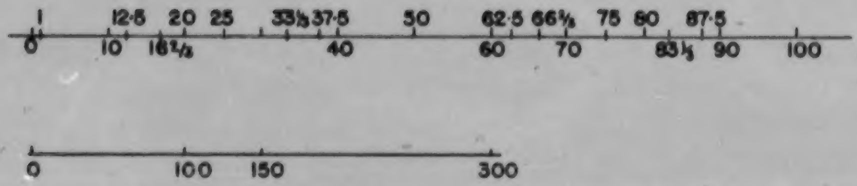
   (a) could be $\frac{4}{6}$, (d) could be $\frac{125.5}{100}$

Answers to Exercises 9-7

1.	(b)	$\frac{25}{100}$	0.25	25%
	(c)	$\frac{3}{4}$	0.75	75%
	(d)	$\frac{1}{5}$	$\frac{20}{100}$	20%
	(e)	$\frac{2}{5}$	$\frac{40}{100}$	0.40
	(f)	$\frac{3}{5}$	0.60	60%
	(g)	$\frac{80}{100}$	0.80	80%

(h)	$\frac{1}{3}$	$\frac{33\frac{1}{3}}{100}$	$33\frac{1}{3}\%$
(i)	$\frac{7}{10}$	0.70	70%
(j)	$\frac{2}{3}$	$\frac{66\frac{2}{3}}{100}$	$66\frac{2}{3}\%$
(k)	$\frac{30}{100}$	0.30	30%
(l)	$\frac{1}{10}$	$\frac{10}{100}$	10%
(m)	$\frac{9}{10}$	$\frac{90}{100}$	0.90
(n)	$\frac{12\frac{1}{2}}{100}$	0.125	$12\frac{1}{2}\%$ or 12.5%
(o)	3	3	300%
(p)	$\frac{3}{8}$	$\frac{37\frac{1}{2}}{100}$	$37\frac{1}{2}\%$ or 37.5%
(q)	$\frac{3}{2}$	$\frac{150}{100}$	1.5
(r)	$\frac{5}{8}$	0.625	62.5% or $62\frac{1}{2}\%$
(s)	$\frac{1}{100}$	$\frac{1}{100}$	1%
(t)	$\frac{87.5}{100}$	0.875	87.5% or $87\frac{1}{2}\%$
(u)	1	$\frac{100}{100}$	1
(v)	$\frac{1}{6}$	$0.16\overline{6}\ldots$	$16\frac{2}{3}\%$
(w)	$\frac{83\frac{1}{3}}{100}$	$0.83\overline{3}\ldots$	$83\frac{1}{3}\%$
(x)	$\frac{11\frac{1}{9}}{100}$	$0.11\overline{1}\ldots$	$11\frac{1}{9}\%$
(y)	$\frac{121}{200}$	0.605	$60\frac{1}{2}\%$
(z)	$\frac{1}{200}$	$\frac{0.5}{100}$	0.5%

2.



4. (a) $\frac{8}{25}$ (b) $\frac{9}{10}$ (c) $\frac{6}{5}$

5. (a) 52% (c) 95%

(b) 35% (d) 30%

The study of percent is valuable if the pupil can use the concept to solve problems from everyday experience, and can understand and interpret data expressed in percent. Pupils are given all three cases of percent from every day situations. The examples of solutions all follow the pattern $\frac{a}{b} = \frac{c}{100}$. In class discussion the estimation of a reasonable answer helps the pupil to understand the relationships among a, b, and c in various situations, as well as serving as a check on an answer.

9-8. Applications of Percent

It is unfortunate that in most texts, percent is considered a separate topic and traditionally receives very special and separate treatment. This separation from the rest of the course is the reason for much of the difficulty encountered. One should not go on to the applications of percent until its relationships with ratio and decimals have been firmly established. Since the solutions are based on the fact:

$$\text{if } \frac{a}{b} = \frac{c}{d} \quad \text{then } ad = bc \quad (b \neq 0, \ d \neq 0)$$

it may be necessary in some classes to review this.

If the background is firm along the lines suggested, there should be little trouble with the applications since terminology is kept at a minimum. The purpose of the section on applications is to show how percentage works, rather than to make the student a facile doer of problems in discount, commission and the like.

Notice that at the beginning of this section, the form $\frac{a}{b} = \frac{c}{d}$ is used exclusively. The reason for this is to stress the essential unity of all problems in percentage, to strengthen the relationship:

$$\text{if } \frac{a}{b} = \frac{c}{d}, \quad \text{then } ad = bc, \quad (b \neq 0, \ d \neq 0)$$

and to prepare for the solution of equations later on. If the pupils discover valid short-cuts for themselves they should be encouraged to use them if they understand what they are doing. But the moment one starts to use special methods for special problems, one is led inexorably to the classification of problems by types, which, as every good teacher knows, must be avoided at all costs.

Of course, one must introduce the ideas of discount, commission and interest one at a time with a little practice on each, but they are mixed as much as feasible so that the pupil will acquire the idea that, though the terms are different, the mathematical processes are the same.

A little space is given to checking by estimation. Some teachers may wish to stress this further. However, even for estimation, it was felt that the solution should follow the same pattern as for the exact determination.

Finally, it should be stressed again that the purpose of the use of applications here is to illustrate the use of percentage. Usage varies from time to time and place to place. If one understands the fundamental meaning of percentage, he should be able to adapt this knowledge to varying usages. It is on this understanding that the stress should be laid.

[pages 382-384]

Answers to Exercises 9-8a

1. (a) 40 (c) 20
   (b) 32 (d) 26
2. (a) 75% (c) 45%
   (b) 90% (d) 95%
3. (a) $0.70 (b) $1.98
4. $875.
5. $180,000.
6. 30%
7. Yes; the two commissions, $6580. and $4820. total $11,400.
8. (a) $4.37 ($5.98 – $1.61)
   (b) $2.48 ($3.40 – $ .92)
9. (a) 1117 (d) 32%
   (b) 34% (e) 100%
   (c) 34%
10. $2400.
11. $60.00, $1060.00
12. 7%, 107%
13. $33\frac{1}{3}\%$, $3\frac{1}{3}\%$
14. 1.2%, $300.00 (It is in problems of this kind that estimation to avoid a mistake in placing the decimal point is especially useful.)

[pages 385-387]

*15. The final net price would be the same for both methods of computation. After a number of examples, most pupils would see that the answer for a price of $200 would be twice that for $100, and similarly for any multiple of $100. If a pupil had studied some algebra it could be shown as follows.

Computing it the first way we would have

$$P - (0.10)P - (0.05)\,[P - (0.10)P] =$$

$$P - (0.10)P - (0.05)\,P + (0.05)(0.10)P$$

Computing it the other way we would have

$$P - (0.05)P - (0.10)\,[P - (0.05)P] =$$

$$P - (0.05)P - (0.10)P + (0.10)(0.05)P.$$

But this would be too much to expect of any except the most exceptional pupils.

*16. If the sales tax were computed on $100, the price would be $102 before the discount, and $91.80 after, just as before. This may seem a little strange since in the latter case, the discount is on the larger amount and the percentage of discount is larger than that of the tax. The same arguments based on examples could be used here, as were used in the previous problem. The algebra would look like this: By the first computation, the final amount would be:

$$P - (0.10)P + (0.02)\,[P - (0.10)P] =$$

$$P - (0.10)P + (0.02)P - (0.02)(0.10)P$$

and by the second it would be

$$P + (0.02)P - (0.10)\,[P + (0.02)P] =$$

$$P + (0.02)P - (0.10)P - (0.10)(0.02)P$$

*17. Under the computations of Problem 15, the net price is $85\frac{1}{2}\%$ of the original price, no matter what the original price is. This means that the discount is really $14\frac{1}{2}\%$ for this way of computing. The shopkeeper might explain that the discounts were to apply successively, that is, one after the other. As we have seen, it does not matter ,which is applied first.

Answers to Exercises 9-8b

1. (a) 11.4%   (d) 22.7%
   (b) 21.0%   (e) 8.0%
   (c) 36.9%   (f) 100.0%; yes
2. 20%
3. 10%
4. 1550 (1240 + 310)
5. 32.1%
6. (a) $\frac{1}{2}\%$ (b) $4,237.50
7. 8.3% or $8\frac{1}{3}\%$
8. Pupils may be interested in comparing percents of increase.
9. (a) Jones .301; Smith .294
   (b) Jones
10. 28.1%
11. A little over 127 pounds. One should expect this since the 15% which is added is computed on a smaller amount than the 15% which is subtracted. If P is the original weight, it is not hard to show algebraically that the final weight is

$$P - (0.15)^2 P.$$

12. A little over 127 pounds, as in the previous problem. Less than the original weight should be expected since the decrease is computed on a larger amount than the increase. Why it is the same as in the previous problem is harder to see except as the working of a few examples convinces the pupil that the answer will be a fixed percent of the initial weight. Algebraically, since it is $P - (0.0225)P$, the net decrease is $2\frac{1}{4}\%$.

The second lesson on percent of increase or decrease emphasizes the two approaches to such a problem. In a problem involving percent of increase, one method finds the actual increase by subtraction, then the increase is expressed as a percent of the earlier quantity. In the second method the percent that the later quantity is of the earlier quantity is computed first. Then 100% is subtracted to give the percent of increase. Pupils need to realize, for example, that an increase of 15% in a quantity always gives a quantity that is 115% of the original; and, also, a decrease of 15% in a quantity always gives a quantity that is 85% of the original. A few minutes of oral work with the class with these ideas would be helpful.

Answers to Exercises 9-8c

1. (a) 150 (b) 120 (c) decrease of 20%
2. 14.3% or $14\frac{2}{7}\%$
3. 5.6% or $5\frac{5}{9}\%$
4. (a) 15 lb. 2 oz. (b) 124.8%
5. 30.9%
6. (a) The 1960 wages are less than the 1958 wages.

   (b) 96%

   Suggestion: Start with 1958 wages of $100 per week. Some pupils will be interested in finding what percent of increase from 1959 to 1960 will bring the wages back to $100. 25%.
7. (a) 15% (b) 17.6%
8. (a) $60 (b) $60 (c) $60

   The object of this problem is to show that though the wording of the problems is different, the mathematics is the same.

9. (a) \$484.50 (b) \$484.50 (c) \$484.50
Again the object is to show that the mathematics is the same in all cases.

10. 5.3%

11. (a) \$677.42 (c) 677 people
(b) \$677.42 (d) \$ 65.10
By this time the pupil might expect the answers to be the same. It will be fine if he can do this with discrimination. Since here (d) is quite different, he may learn to proceed with caution. The teacher may want to give other exercises like this.

12. (a) 2,258 gal.
(b) 67,742 mi.
(c) About 186 miles per day.
(d) It is unlikely that a man who walks to work would drive 186 miles a day.

Sample Questions for Chapter 9

I. True-False Questions

(T) 1. When 15.86 is rounded to the nearest tenth, it is written 15.9.

(T) 2. One hundred per cent of a number equals the number.

(F) 3. $.09 > .10$

(F) 4. 25% may be represented as $\frac{1}{5}$.

(T) 5. Any rational number may be written in decimal notation.

(F) 6. If one-third of a class is girls, the boys make up 60% of it.

(F) 7. The sum of three tenths and three hundredths is three thousandths.

(T) 8. If $\frac{x}{3} = \frac{60}{15}$, then $x = 12$.

(T) 9. .01 divided by .01 equals 1.

(F) 10. $\frac{1}{10^5}$ (one over ten to the fifth power) may be written as .0001.

(F) 11. If Bob weighs half as much as his father, the ratio showing the comparison of Bob's weight to his father's weight is 2:1.

(T) 12. $\frac{8}{25}$ is another name for the number 32%.

(T) 13. An increase in the price of an item from \$20 to \$28 is an increase of 40%.

(F) 14. If a class has a total of 32 pupils, 20 of them boys, the number of boys is 60% of the number of pupils in the class.

(T) 15. Five percent of \$150 is the same amount of money as 7.5% of \$100.

(F) 16. 62.5% and $\frac{5}{8}$ are names for two different numbers.

II. Matching

Choose items from Column B which make the statements in Column A correct. Place the letters of your choices from Column B in the spaces to the left in Column A.

Column A

b 1. The digit 9 occupies the____ place in the decimal numeral 3284.569.

d 2. If a and b are two numbers and $b \neq 0$, the____of a to b is the quotient $\frac{a}{b}$.

a 3. In the decimal numeral 9384.562 the digit 9 occupies the____ place.

j 4. The decimal numeral 473.45 rounded to the nearest tenth is____.

g 5. The decimal numeral for the rational number $\frac{1}{8}$ is a____ decimal.

h 6. 2.54 cm. equals____millimeters.

k 7. The decimal numeral for the rational number $\frac{2}{7}$ repeats____.

e 8. Another name for the number one is____.

l 9. A____is a statement of equality of two ratios.

m 10. The number $0.45\overline{45}\ldots$ is____ times the number $0.09\overline{09}\ldots$.

Column B

(a) thousands
(b) thousandths
(c) percent
(d) ratio
(e) 100%
(f) 473.5
(g) repeating
(h) 25.4
(i) 254
(j) 473.4
(k) a block of six digits
(l) proportion
(m) 5

III. Multiple Choice

1. Six percent of $350 is

(a) $210.00
(b) $ 21.00
(c) $ 2.10
(d) $2100.
(e) None of these

1. b

2. If 8% of the number 5400 is computed, the correct answer is

(a) More than 30 but less than 90.
(b) More than 3 but less than 5.
(c) More than 45 but less than 450.
(d) More than 500 but less than 1000.
(e) None of these.

2. c

3. If ½% of $320.00 is computed, the answer is

(a) $ 16
(b) $160
(c) $3.20
(d) $1.60
(e) None of these

3. d

4. In a class of 42 pupils there are 25 boys. The number of boys is what percent (nearest whole percent) of the number of pupils?

(a) 60%
(b) 59%
(c) 58%
(d) 61%
(e) None of these

4. a

5. In the class of 42 pupils there are 17 girls. The number of girls is what percent (rounded to the nearest tenth of a percent) of the number of pupils?

(a) 40.4%
(b) 40.6%
(c) 39.9%
(d) 40.7%
(e) None of these

5. e

6. Which of the following repeat a single digit when written as decimals?

(a) $\frac{1}{3}$ (d) $\frac{3}{11}$

(b) $\frac{5}{13}$ (e) $\frac{11}{7}$

(c) $\frac{2}{13}$ 6. a

7. An agent is paid 5% commission. If he sells $1500 worth of merchandise, he should receive

(a) $750 (d) $.75

(b) $75 (e) None of the answers is correct.

(c) $7.50 7. b

8. A ball club won 4 of the 8 games already played. If it wins the next two games, what percent of the games will it then have won?

(a) 80% (d) 50%

(b) 70% (e) 40%

(c) 60% 8. c

9. If Tom was successful in 13 out of 20 tries in practicing free throws, which of the following represents his accuracy?

(a) 87% (d) 20%

(b) 65% (e) 13%

(c) 33% 9. b

10. The product of 046 × .3 is

(a) 0.0138 (d) 13.8000

(b) 0.1380 (e) 138.0000

(c) 1.3800 10. a

11. The quotient 1.44 is the answer to the division problem

(a) $3.6\,\overline{)518.4}$ (d) $.36\,\overline{).5184}$

(b) $3.6\,\overline{).5184}$ (e) $36\,\overline{)518.4}$

(c) $.36\,\overline{)51.84}$

11. d

12. What is another way of writing .065?

(a) 6.5% (c) .6$\frac{1}{2}$%

(b) 65% (d) .65%

12. a

13. What is another way of writing 15%?

(a) $\frac{15}{10}$ (c) $\frac{15}{1000}$

(b) $\frac{15}{100}$ (d) $\frac{15}{1}$

13. b

14. John sells magazines and may keep 20% of the money he collects. If his sales are \$3.50 he may keep:

(a) \$ .50 (d) \$3.50

(b) \$ .70 (e) None of the answers is correct.

(c) \$3.00

14. b

IV. Problems

1. What commission will a real estate agent receive for selling a house for \$15,400 if his rate of commission is 5%? \$770

2. The sale price on a dress is \$22.80 and the marked price showing on the price tag is \$30.00. What was the rate of discount? 24%

3. An increase in rent of 5% of the present rent will add \$3.50 to the monthly rent that Mr. Johnson will pay. What is the monthly rent that Mr. Johnson now pays? \$70

4. A family budget allows 30% of the family income for food. If the monthly income of the family is \$423, what amount of money is allowed for food for the month? $\underline{\text{\$126.90}}$

5. Dorothy was 5 feet tall (to the nearest inch) when school opened in September. In June her height was 5 feet 3 inches. What is the percent of increase in her height? $\underline{5\%}$

6. (a) Write $4(1) + 3\left(\frac{1}{10}\right) + 5\left(\frac{1}{10^3}\right)$ in positional notation. $\underline{4.305}$

   (b) Write 0.12 in expanded form. $\underline{1\left(\frac{1}{10}\right) + 2\left(\frac{1}{10^2}\right)}$

7. Find the decimal to the nearest tenth for $\frac{17}{38}$. $\underline{0.4}$

8. (a) Round 14.657 to the nearest tenth. $\underline{14.7}$

   (b) Round 5.9349 to the nearest hundredth. $\underline{5.93}$

9. Find the decimal to the nearest hundredth for $\frac{21}{13}$. $\underline{1.62}$

10. At a certain time of the day a man 6 feet tall casts a shadow 8 feet long. At the same time a nearby tree casts a shadow of 40 feet. How tall is the tree? $\underline{\text{30 feet}}$

11. What is the greatest possible error in a measurement that is made to the nearest centimeter? $\underline{\text{one-half centimeter}}$

12. How much is the difference of the sum of 1.05 and 0.75 and the sum of 0.5 and 0.125? $\underline{1.175}$

13. The decimal numeral for the rational number $\frac{1}{11}$ is____. $\underline{0.0\overline{909}\ldots}$

14. Calculate the measure (in cm.) of the perimeter of a square whose edges measure 30 cm. 120

15. Calculate the area (in sq. cm.) of the interior of a square whose edges measure 30 cm. 900 sq. cm.

16. Calculate the volume (in cu. cm.) of the interior of a cube whose edges measure 30 cm. 27,000 cu. cm.

17. If Bob's shadow is 80 inches long when John's shadow is 120 inches long, how tall is John if Bob is 30 inches tall? 45 in.

18. A cookie recipe using $1\frac{1}{2}$ cups of flour will yield 48 cookies. If Janet wants to bake 24 cookies, how much flour should she use? $\frac{3}{4}$ cup

CHAPTER 10

PARALLELS, PARALLELOGRAMS, TRIANGLES, AND RIGHT PRISMS

"Informal geometry" as presented in this chapter, is concerned with the discovery of geometric principles through experimentation and, where feasible, the verification of empirical conclusions by deductive reasoning. Students perform as scientists in collecting data and then perform as mathematicians in the analysis and interpretation of the data they have obtained. Data needed to formulate a statement of a geometric property are obtained by measurement, with protractor or ruler, or by superimposing one figure on another.

Pupils are introduced to the use of deductive reasoning as a method for ascertaining what is true about a geometric figure, arguing from previously stated principles and definitions. We reserve for a later time the systematic organization of geometry as a deductive system, starting with postulates and undefined terms, and developing theorems and definitions on this basis.

The specific purposes for which this chapter were planned are these:

1. To develop awareness of the ideas of points, lines, and planes, and their intersections.
2. To give the pupils experience in verification of experimental results by informal deductive argument on the basis of previously stated principles.
3. To introduce certain geometric concepts and relations as listed below.

The major topics are:

1. Some angle relationships in the figure formed by parallel lines and their transversal.
2. The angle and side relationships in a triangle.

3. The angle and side relationships in a parallelogram.
4. Areas of parallelograms and triangles.
5. Volumes of right prisms.

Some General Observations and Suggestions

As in other chapters, precise terminology is emphasized throughout the text material. It is necessary to make this emphasis because many of the words that are casually used by seventh graders are not as clearly understood by their users as we would like to have them. At this level, the consequences of casual language are not always serious but may become so as students proceed in their mathematical studies. All of the terminology developed in previous chapters should be used whenever such usage clarifies and simplifies geometric statements but care should be exercised that in our attempts to be as exact as possible, we do not make a complicated thing out of what may be, essentially, a very simple idea. To avoid this situation, it is suggested that meanings be given first in words of common usage and then in the more precise terminology. The translation from common usage to precise usage then becomes an exercise in analytical thinking.

Ideally, once a word that is commonly used is pre-empted for a special meaning in a new vocabulary, the new meaning must be adhered to from that point on. In actual fact, however, it is often difficult to convince a seventh grader that he should do this. In this case we should accept, for the time being, his way of speaking, evaluate and discuss what ideas he is attempting to present, and then encourage him to rephrase his statements in more precise words.

Since students often learn best by imitation and habit formation, it is suggested that the teacher become thoroughly familiar with the new terminology and use it at every possible occasion. Through the simultaneous use of both the common and the precise ways of speaking it is hoped the student will become more and more proficient in the latter and grow to appreciate its value until he eventually uses it as a matter of course.

Some particular terminology used in the text may need further clarification. "Intersect in the empty set" and "are parallel" as applied to lines have the same meaning and may be used interchangeably where the lines are in the same plane. "Intersect in the empty set" or "are skew" have the same meaning when applied to lines that are not in the same plane. The front edge of the ceiling and the side edge of the floor are skew and have no point of intersection. Which of these phrases to use in a particular context depends on the conclusions one wishes to draw. If the question is, "What are all the possible intersections of two lines in space?" one of the possible intersections is "the empty set." This is a phrase that might seem preferable to "parallel lines have no intersection." On the other hand, if the question is "How are the opposite sides of a parallelogram related?" then "they are parallel" might seem preferable to "they intersect in the empty set." Use your own judgment in matters of this kind. It may be noted that the authors at times say "the lines do not intersect" even though this is not strictly so in the precise set terminology. We say "the lines may intersect in the empty set." Since the phrase "do not intersect" is so commonly used, it seems desirable to use it here. It is also thought that there is some advantage in presenting some ideas in more general terms, which set language permits.

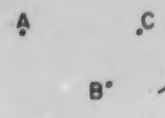
The chapter includes a few deductive developments of a more or less informal nature. One of the problems arising in such a development is that pupils usually fail to appreciate the need for justifying statements with reasons previously adjudged acceptable to the group as a whole. One proposal that might impress them with the fact that only previously stated and accepted properties, definitions, and reasons should be used is to suggest that football and basketball games would be much more confusing if in each game the rules were changed without consulting anybody and that new rules be made up as the game goes along! It might be an interesting game but hardly a fair one!

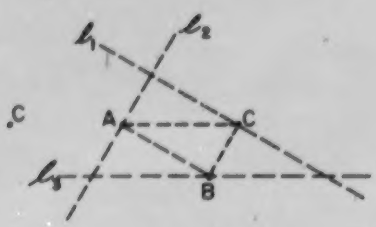
An occasional reminder about "making up rules as you go along" is usually sufficient to make the point desired.

Frequently students are asked to make a general statement about a property or the results obtained through experiment. In the text such statements are partially written so that the grammatical form of the statement is suggested without hinting too strongly at the mathematical ideas involved. Before considering these, students should be encouraged to formulate their own statements of principles and properties but such statements should be very closely examined to ensure that the meaning is precise and clear. When a statement seems satisfactory to all, then show pupils the formulation in the answer section for comparison. They then may use these statements as models in future work in this chapter.

About a week prior to the introduction of this chapter, pictures and articles pertaining to plane geometry could be placed on the classroom bulletin board to arouse curiosity and supplement the historical facts briefly mentioned in the first section. Here are two puzzles that will add interest to the display:

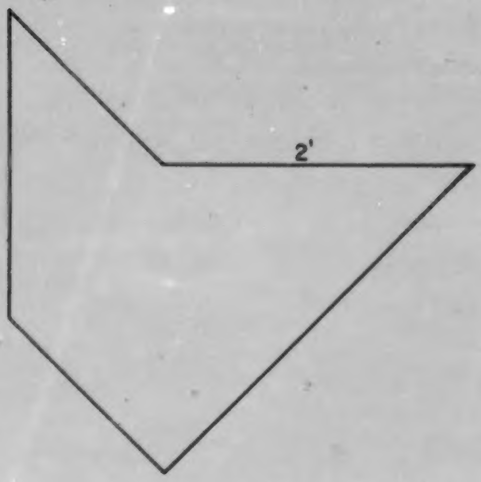
1. Drawing a triangle is easy but can you draw a triangle so that each of the dots lettered A, B, and C are midpoints of its sides?

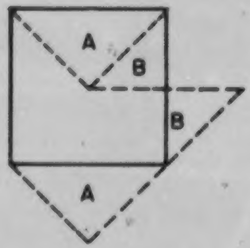




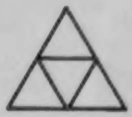
$\ell_1 \parallel \overline{AB}$
$\ell_2 \parallel \overline{BC}$
$\ell_3 \parallel \overline{AC}$

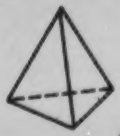
2. A very thrifty cabinet-maker wished to construct a table top two feet square out of a piece of plywood shaped as in the figure. He was able to do this with only two sawings. If you are as clever as the cabinet-maker, you can do the same.





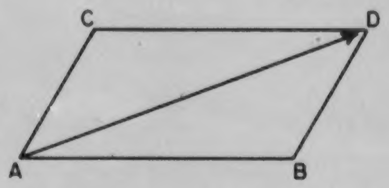
Pictures and geometric diagrams suitable for posting are not always easy to find but some good ones can be obtained from such magazines as *Scientific American* , *Fortune* , *Popular Science* , *Popular Mechanics* , *Life* , and, of course, some technical magazines. A pattern of "oak tag" paper might be posted along with a completed model for the construction of a regular tetrahedron as an activity in equilateral triangles.





The following hints are given about applications for the purpose of motivation. These may be supplemented by material found in the introductory chapters of plane geometry texts that are available.

There is never an over-supply of good thinkers. The world needs people who can begin with a body of facts, relate them, and think through to logical conclusions. The study of geometry helps to train such people. In the aircraft industry there is a great demand for workers trained in geometry because there is a considerable amount of geometric knowledge involved in the construction of an airplane. The main problem is to learn how air will flow about an airplane of given shape moving in a given direction at a given speed. From this the lifting force and the air resistance may be calculated. The parallelogram of forces may be used for an illustration. In order to find the single force equivalent to two forces acting simultaneously at a point we can draw a diagram like this in which the given forces are represented in magnitude and direction by the segments $\overline{AB}$ and $\overline{AC}$. We complete the parallelogram, and the diagonal $\overline{AD}$ gives the magnitude and direction of the resultant force.



Geometry is also used to determine the forces in an electromagnetic field, and why rubber is elastic, and how an oil company should schedule its production. In the theory of relativity and in the design of agricultural experiments completely different concepts of space are used. Today, the physicist, the chemist, the biologist, the engineer, the economist, the psychologist, and the military strategist use geometry in ways far removed from some of those which were not even discussed or dreamed of only fifteen years ago.

10-1. Vertical and Adjacent Angles

Concepts to be developed:

1. When the intersection of two lines is not the empty set, four angles and four half-planes are determined.
2. Two angles which have a common ray and whose interiors have no points in common are called adjacent angles.

3. Non-adjacent angles determined by the intersection of two lines [in a single point] are called vertical angles.
4. When two lines intersect in a point, they determine four different half-planes. Two of these half-planes intersect in the interior of one angle in a pair of vertical angles. The interior of the other angle in the pair of vertical angles is the intersection of the remaining two half-planes.
5. The angles in a pair of vertical angles are congruent, that is, are equal in measure.

Suggestions

Review very briefly the possible intersections of two lines in space using two meter sticks, pointers, or pieces of coat-hanger wire to represent a pair of lines intersecting in space. If the intersection is not the empty set, both wires should be grasped in one hand at their point of intersection so that the other hand is free to indicate parts of the figure. Use this same device to suggest adjacent angles and vertical angles. By this procedure, students will be encouraged to think of these ideas in terms of "general" space and not just that portion of space represented by the chalkboard or paper. Also, it is often more convenient and more time-saving to carry a geometric figure to the students in this way than to have the students carry themselves to the same figure drawn on the chalkboard.

Pupils may say "Angle A is a vertical angle". This is not good terminology because we always refer to vertical angles in pairs. A similar remark holds for adjacent angles. "Adjacent angles" should be used only in a phrase which includes some mention of a pair of angles.

In Problem 5(e) it is not necessary to formalize the subtraction axiom. The notions involved in the deduction required should already be understood and accepted by the group from previous experience with number sentences.

In part (e) of Problem 6, the reason for m(∠ x) = m(∠ z) should be developed as a statement of the equality of two numbers and that this particular equality is true because m(∠ x) and m(∠ z) are "two names for the same number."

Answers to Questions in Section 10-1

Yes. The half-plane containing point D intersects the half-plane containing point C to form the interior of ∠ CAD.

Yes. The half-plane containing B intersects the half-plane containing D to form the interior of angle BAD.

Answers to Exercises 10-1

1. (a) ∠ JKL, ∠ MKN
   (b) ∠ JKL, ∠ MKN
   (c) ∠ JKM, ∠ LKN
2. (a) ∠ LKN
   (b) ∠ JKL and ∠ MKN
   (c) Two pairs of vertical angles are formed.
3. (a) 40 = m(∠ JKM) 40 = m(∠ LKN)
   (b) 140 = m(∠ NKM) 140 = m(∠ JKL)
   (c) The angles of a pair of vertical angles seem to have equal measures.
   (d) They do appear equal.
4. Property 1: When two lines intersect, the two angles in each pair of vertical angles which are formed have equal measure, or are congruent.

*5. (a) 180 (c) 120, 120

(b) 180 (d) 110, 110

(e) 180 = sum of $m(\angle JKM)$ and $m(\angle JKL)$
180 = sum of $m(\angle JKM)$ and $m(\angle NKM)$
Then $m(\angle JKL) = m(\angle NKM)$ since they both must be names for the same number.

6. (a) 180

(b) 180

(c) Subtract $m(\angle y)$ from 180. Subtract $m(\angle z)$ from 180.

(d) $m(\angle x) = 180 - m(\angle y)$;
$m(\angle z) = 180 - m(\angle y)$

(e) $m(\angle x) = m(\angle z)$

7. (a) Yes, two lines which are skew.

(b) The edge of the floor at the front of the room and the edge of the ceiling at one side.

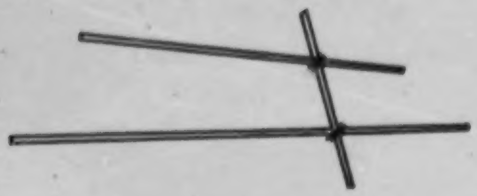
10-2. Three Lines in a Plane

Concepts to be developed:

1. A line which intersects two or more lines in distinct points is called a transversal.
2. In any figure consisting of lines whose intersections are not empty, there are at least two pairs of vertical angles and at least four pairs of adjacent angles.
3. Two angles the sum of whose measures is 180 degrees are called supplementary angles.
4. Supplementary angles are not necessarily adjacent.
5. Two different angles whose interiors lie on the same side of a transversal such that a complete ray of one is contained in a ray of the other, are called corresponding angles.

[pages 402-403, 404]

It will prove helpful to make a demonstration model constructed of three flat sticks free to turn on pivoted connections made with rubber bands, small round-head bolts, or of clinched tacks or nails.



On the chalkboard represent parallel lines in other than horizontal positions at first. Otherwise, students get the habit of thinking of parallel lines as being horizontal and have difficulty seeing them in complicated figures where the parallel lines are not horizontal. Colored chalk lends interest to a chalkboard drawing and is effective in calling special attention to certain portions of a figure.

Pupils should draw figures according to the directions in the exercises even though the figures appear in the text. This activity helps to fix the features of a drawing in their minds. Students' drawings frequently are too small. In the exercises of this section lines should be represented by segments about two to three inches long. Free-hand sketches are acceptable if done carefully.

Problem 6 of the exercises is of great importance and by no means should be omitted. The basic definition included in this problem requires that it be done by all pupils.

Answers to Questions in Section 10-2.

Yes; yes; no.

Answers to Exercises 10-2.

1. (a) Four pairs

(b) Eight pairs

(c) Yes

(d) No. Since t_1 and t_2 are parallel they do not intersect. To be a transversal of t_2 and t_3, line t_1 must intersect both lines.

2. (a) Triangle R Q S

(b) Six pairs

(c) Twelve pairs

3. (a) Line t is the transversal and intersects ℓ_1 and ℓ_2. If the lines are extended any line may be considered as a transversal of the other two.

(b) Eight angles; twelve if ℓ_1 and ℓ_2 intersect.

(c) Four pairs; six pairs if ℓ_1 and ℓ_2 intersect.

(d) The angles in any pair of vertical angles have equal measures.

(e) Yes, because $\angle h$ and $\angle f$ are vertical angles.

(f) No, adjacent angles may or may not be congruent.

4. (a) Eight pairs; 12 if you consider the intersection of ℓ_1 and ℓ_2.

(b) Sum of measures is 180.

5. (a) Yes

(b) No

(c) Yes (Neither straight nor reflex angle has been defined.)

(d) $m(\angle g) = 100$ $m(\angle e) = 100$ $m(\angle f) = 80$

(e) Yes

(f) Yes

6. (a) ∠ d and ∠ g
   (b) Yes
   (c) No
   (d) Four pairs
   (e) No
   (f) The measure of each angle is 90.
   (g) Yes

*7. (a) Each shows two lines with their transversal. (There are others.)

   (b) The figures differ in the number of pairs of lines whose intersection consists of one point. (There are others.)

   (c) All three lines may be parallel.

   (d) The intersections of three different lines in the same plane may consist of 0, 1, 2, or 3 points.

*8. With three planes in space the cases are:

   (a) All pass through a point. (The corner where two walls and the ceiling meet.)

   (b) Two planes parallel but both intersected by the third (floor, ceiling and one wall).

   (c) Each pair of planes intersect but their lines of intersection are parallel. (In one type of house, the planes of the roof and the plane of the attic floor.)

   (d) Three parallel planes (a stack of shelves).

10-3. Parallel Lines and Corresponding Angles

Concepts to be developed:

1. When in the same plane a transversal intersects two lines and the corresponding angles have unequal measures, then the two lines will intersect.

2. When in the same plane a transversal intersects two lines and the corresponding angles have equal measures, then the two lines do not intersect.

Suggestions

The emphasis should be on the children's discovering the relationships by their own observations. Be careful not to "kill" their interest by telling them the answers. Ask leading questions and try to draw the answers out of the students.

If class time is short, the experiments in this section may be speeded up if each student is assigned the measuring for two cases and then reports his findings for tabulation on the chalkboard. It might be well to carry out other experiments in this same manner.

If an overhead projector is available, by all means use it for this section. On one piece of plastic film, draw with china-marking pencil line r_2 and transversal t. On another piece draw line r_1. By superimposing the second piece on the first and projecting the image on the chalkboard, line r_1 can be rotated to different positions through Point A and observations as to angle measures and intersections can be tabulated on the chalkboard. Since the figure is projected on the chalkboard, marks may be erased without obliterating the figure.

Answers to Class Exercises and Discussion 10-3

1. Yes, they intersect on the left side of t.
2. Yes, r_1 and r_2 intersect on the right side of t.
3. Third column, third entry is "the empty set." If the corresponding angles are unequal, r_1 and r_2 intersect in a point to the right or left of the transversal t. If the corresponding angles are equal, r_1 and r_2 are parallel.

4.

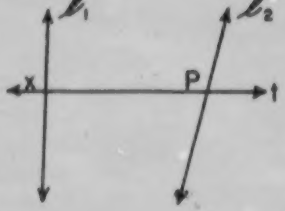
Intersection of r_1 and r_2
right side
empty set
left side

5. measures, intersect

6. parallel

Answers to Exercises 10-3.

1.



Intersection of 1 and 2
(a) below
(b) parallel
(c) above
(d) below
(e) parallel
(f) above
(g) below
(h) parallel
(i) above

2. As in classroom (opposite edges of ceiling, etc.).
3. 90. Minimum distance should occur when the transversal is perpendicular to the lines. (This idea is more fully developed in Section 9-7.)

10-4. Converses (Turning a Statement Around)

Concepts to be developed:

1. A converse of an "if -- then" sentence may be formed by interchanging the "if" part and the "then" part.
2. A converse of a statement may be true or false regardless of whether the original statement is true or false.

Suggestions

Caution students that the meanings of words used in the sample statements of the text are understood to be those in the widest use. "Mary and Sue" are not names for ships!

Care must be taken not to refer to a converse as the converse because for any one statement there may be several different converses. To form a converse it is only necessary to exchange one fact of the "if" clause with one fact of the "then" clause and where these clauses contain more than one fact, there will be several converses. It is felt that the "turning around" device is as far as one needs to go for seventh graders.

It might prove helpful to clarify the meaning of "true" or "false" before tackling the exercises. Without going into details of logic, it should be sufficient to propose that a "false" statement is one that is not always true and a statement is not considered true if at least one counter-example can be found. For instance, let us examine the statement: "The set of whole numbers is closed under subtraction." As a counter-example one might say, "There is no whole number which added to five gives three and, therefore, (3 - 5) is not a name for a whole number." This one counter-example is all that is needed to deny the quotation. Emphasize that only one counter-example is needed to prove a statement false.

While one counter-example can be used to show that a statement is false, it is a great deal more difficult to show that a statement is true. In this section we do not expect that the students or teacher will prove that statements are true. In Section 6 a geometric proof is given for the property of the sum of the angles of a triangle, and it is suggested in the exercises that students try to discover proofs for several other properties. In this section only inductive arguments can be given to indicate that a statement is true. Students should be reminded that even though thousands of cases were investigated and all were in agreement with the statement or property, still the property would not necessarily be true. Also, failure to find a counter-example will not constitute a proof, since it is always conceivable that someone else might find such an example.

In "if -- then" statements, the "if" clause is considered to be the postulate or postulates. In mathematics it is recognized that even a formal proof is based on postulates and that the property to be proved is true only if the postulates used in the proof are true. Mathematicians do not recognize any absolute truth in mathematics. A mathematical agreement, based on postulates which may not necessarily even be in agreement with experience, is considered a proof, provided deduction is used correctly. Again, it is recognized that the validity of the proof does not determine the truth of the property. The truth of a property is completely dependent on the truth of the postulates.

Answers to Questions in Section 10-4

(b) Not always

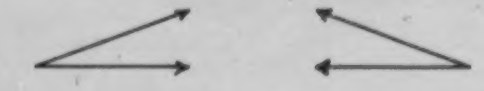
1. Yes; No

2. Yes; Yes

3. Statement "a" is true.
   Statement "b" is false.

Answers to Exercises 10-4

1. One of many examples is:



No.

2. (a) false
   (b) true
   (c) true
   (d) true (if no amputees)
   (e) true
   (f) true

3. (a) If Blackie is a cocker-spaniel, then Blackie is a dog. (true)
   (b) If it is a holiday in the U.S., then it is July 4th. (false)
   (c) If Robert is the tallest boy in his class, then he is the tallest boy in his school. (false)
   (d) If an animal has four legs, then the animal is a horse. (false)
   (e) If an animal has thick fur, then the animal is a bear. (false)
   (f) If Mark is Susan's brother, then Susan is Mark's sister. (true)

4. (a) true
   (b) true
   (c) false
   (d) true
   (e) false
   (f) true

5. (a) If two lines are parallel and are intersected by a transversal, a pair of corresponding angles are congruent.
   (b) The angles are congruent. Classmates should have similar results.
   (c) True

6. If in a plane, a transversal intersects two lines and the two lines intersect in a point, then the angles in a pair of corresponding angles formed are unequal in measure. (true)

7. Yes. If a figure is a simple closed curve, then it is a circle. (false)
If a figure is a circle, then it is a simple closed curve. (true)

10-5. Triangles

Concepts to be developed:

1. A set is determined in accordance with a common property.
2. There are three sets of triangles determined according to the measures of their sides.
   (a) The set of isosceles triangles has as members triangles which have two sides that are equal in length.
   (b) The set of scalene triangles is the set of triangles in which no two sides have the same measure.
   (c) The set of equilateral triangles is the set of triangles which have three sides equal in length.
3. An angle and a side of a triangle are said to be opposite each other if their intersection contains just the endpoints of the segment referred to as side.
4. If two sides of a triangle are equal in length, the angles opposite these sides have equal measures.
5. If two angles of a triangle have equal measures, then the sides opposite these angles are equal in length.

Suggestions

Remind students that a set should be "well-defined." There must be no doubt or question as to what is or is not a member of a set. This does not mean, however, that a definition of set should be attempted. The set of beautiful paintings in the world is not a "well-defined" set because what is beautiful is controversial. The set of triangles is not a "well-defined" set if it is not clear whether spherical triangles are to be included. In our text, since all triangles are understood to be plane figures the set of triangles is clearly defined. In other words a set is well-defined if and only if there is no doubt as to whether or not a thing belongs in the set. This idea is important in defining the set of isosceles triangles and emphasizes the need for precise and complete terminology.

Students should try to answer questions in the text as they reach them and not read ahead for the answers. Answers are included in the text so that, if desired, all or a portion of the text may be assigned for reading outside of class.

Have a number of soda straws measured and creased before class begins. This exercise may seem easy to most pupils but even the brighter ones will jump to incorrect conclusions about a straw divided into three pieces of 2", 3", and 5". Soda-straw figures give the class an opportunity to handle triangles in regions of space other than the chalkboard or the drawing paper. Make certain that the soda-straw figures are understood to be merely representations of triangles and are not actual triangles.

Answers to Questions in Section 10-5

Figures (b) and (f) are triangles.

(a) Segment joining two points is missing.

(c) and (d) Four points instead of three.

(e) Two points joined by a curve which is not a straight line segment.

Common property: At least two sides are equal in length.
Common property: All three sides have equal measures
No; no; no

Soda-straw figures.

(a) scalene

(b) equilateral and isosceles

(c) isosceles

(d) isosceles

(e) equilateral and isosceles

Concerning first drawing (Triangle ABC):
Yes, because A and C are the only points shared by angle B and side $\overline{AC}$.
No, because points A and C are not the only points shared by angle C and $\overline{AC}$.
Angle C and side $\overline{AB}$, because points A and B are the only points shared by angle C and side $\overline{AB}$.
Concerning second drawing (Triangle ABC):
Isosceles, because at least two sides have equal measures.
They are equal.
The measures of at least two angles in each triangle are equal.
Property 3: If two sides of a triangle are equal in length, then the angles opposite these sides have equal measures.
Concerning third drawing (Triangle ABE):
They are equal. The sides opposite the equal angles are equal.
Isosceles triangles.
If two angles of a triangle are equal in measure, then the sides opposite these angles are equal in length.
Yes. They are converses.

Answers to Class Discussion Problems 10-5

1. Yes. Advantages:

   (1) Useful when only angle measures are known.

   (2) Ruler may not be available.

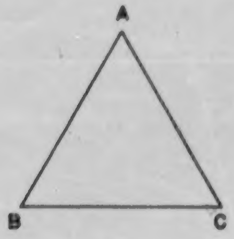
   Disadvantages:

   (1) Protractors not as generally available as rulers.

   (2) Angle measure is often more difficult to estimate than linear measure.

2.

1.	$m(\angle A) = m(\angle B)$	1.	Given
2.	$m(\overline{BC}) = m(\overline{AC})$	2.	Converse of Property 3
3.	$m(\angle C) = m(\angle B)$	3.	Given
4.	$m(\overline{AB}) = m(\overline{AC})$	4.	Converse of Property 3
5.	$m(\overline{BC}) = m(\overline{AC})$ $= m(\overline{AB})$	5.	All names for the same number.



Answers to Exercises 10-5

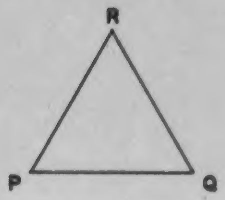
4. Not necessarily, because an equilateral triangle must have all three sides equal and an isosceles triangle does not satisfy this requirement.

5. Yes, because an equilateral triangle does satisfy the requirements for being isosceles by having two sides of the same length. "Only two sides equal in length" is not part of the requirement for isosceles triangle.

6. P opposite $\overline{QR}$

   Q opposite $\overline{PR}$

   R opposite $\overline{PQ}$



7. (a) No. The outside two sections, even if held end to end in a straight line, would just reach from one end of the 3-inch section to the other. Thus no triangle would be formed.

   (b) No. If the end pieces were folded toward each other, they would not meet.

   (c) The sum of the measures of any two sides of a triangle must be greater than the measure of the third side.

10-6. Angles of a Triangle

Concepts to be developed:

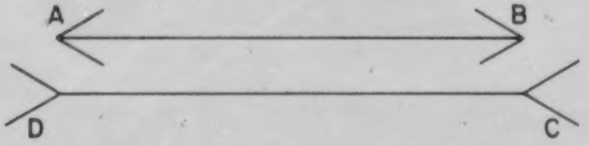
1. The sum of the measures of the angles of a triangle in a plane is 180.
2. A property may be suggested by inductive reasoning but it is proved by deductive reasoning.
3. The proof of Property 4 applies to any triangle in a plane.

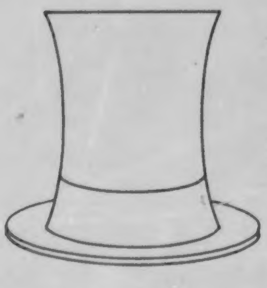
Suggestions

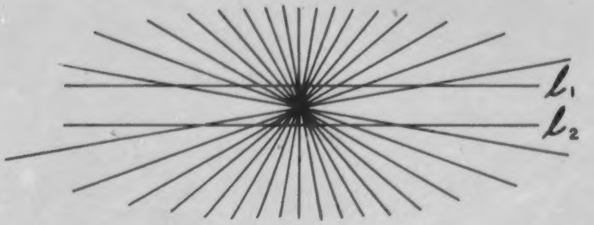
Since the class exercises require the folding and cutting of triangular shapes, much class time may be saved by supplying the class with a quantity of such shapes already cut out instead of spending time in the distribution and collecting of scissors. By using a paper cutter and cutting several pieces of paper at a time, a sufficient supply can be easily obtained. Triangles should be fairly large and as different as possible.

Whether the mounting and pasting suggested in the text is done in class or not is optional. In fact, this particular activity could be a good "lesson clincher" if made a part of the homework assignment. However, the tearing and folding for the purposes of experiment should be done in class as suggested.

The angle sum for a triangle is first developed experimentally and then proved deductively. To some pupils it may seem silly to prove deductively what is perfectly obvious intuitively. One way to meet this criticism is to shake these students' faith in the infallibility of their intuition. Here are a few optical illusions, quickly and easily drawn on the chalkboard, that might do the trick:







$\overline{AB}$ and $\overline{CD}$ have the same length. ℓ_1 and ℓ_2 do not bulge apart. The top hat is as wide as it is tall.

In this paragraph are mentioned other ways helpful in emphasizing the need for verification of "obvious" or intuitive conclusions. Test students' ability to distinguish between two sharp points by touch alone (try this on finger tips first and then on the back of the neck), or ask what the bill would be for shoeing a horse at one penny for the first nail, two for the second, four for the third, eight for the fourth, etc. for four shoes with eight nails to the shoe. Tell about the native of China who for seventy years, upon awakening in the morning, noted that the first person he saw was another Chinese. On the eve of his seventy-first birthday, after having made the same observation approximately 25,915 times, he went to sleep certain in the knowledge that the first person he would see in the morning would be another Chinese. It was a Russian!

Spend some time with the class discussing why the deductive proof applies to all planar triangles and why the conclusions drawn from empirical data in Exercises 1 and 2 apply only to the particular triangles tested.

It is recognized that Problem 4 of the classroom exercises will be difficult, particularly the attempt to decide on a legitimate reason for each step. This is the only example of a formal deductive proof given in the chapter, however, and it is hoped that the teacher will go through it with the class, though not expecting real mastery. It might be helpful to recall for the class the idea of "making up the rules after the game has started" as suggested in the second paragraph, Page 4 of this commentary.

Answers to Questions in Section 10-6

1. (a) 180, yes
   (b) The sum of the measures of $\angle 1$, $\angle 2$, and $\angle BAC$ is 180.
2. (e) the sum appears to be 180
   (f) Yes (g) Yes
3. (a) Yes, because of Property 2a.
   (b) Corresponding angles. Yes. Converse of Property 2a.
   (c) Vertical angles. Yes. Property 1.
   (d) Were drawn so as to have equal measures.
   (e) The measures in the sum on the left are equal to the measures in the sum on the right.
   (f) By definition of "sum."

(g) Angles on one side of a line.

(h) Two names for the same number as indicated in steps (e) and (g).

(i) The translation of the symbols in step (h) into words.

Answers to Exercises 10-6.

1. 60

2. (a) 60 (c) 5
   (b) 30 (d) 20

3. Yes, either 65 and 65, or 50 and 80.

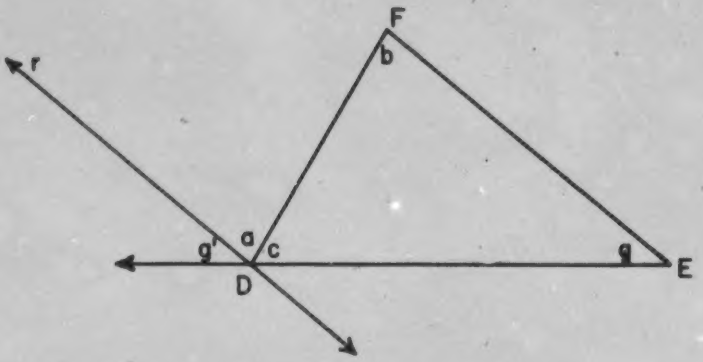
4. m($\angle$ ABC) = m($\angle$ DEF) Property 4.

5. (a) 80 Property 4 (d) 60 Property 3 and 4
   (b) 91 Property 4 (e) 4 Property 4 and converse Property 3.
   (c) 40 Property 3

6. (a) Yes, Property 1
   (b) Yes, Property 2 or converse of Property 2a
   (c)

1. m($\angle$ y) = m($\angle$ n)	1. Property 1
2. m($\angle$ y) = m($\angle$ u)	2. Property 2 or converse of Property 2a
3. m($\angle$ n) = m($\angle$ u)	3. Two names for the same number, m($\angle$ y).

7.



Assign letters to other angles of the figure than these to which letters have already been assigned.

1.	$m(\angle g') = m(\angle g)$	1.	As drawn
2.	line r is parallel to $\overline{FE}$	2.	Property 2a
3.	$m(\angle b) = m(\angle a)$	3.	Property of Problem 6
4.	$m(\angle g') + m(\angle a) + m(\angle c) = 180$	4.	Angles on one side of a line.

5. $m(\angle g') + m(\angle a) + m(\angle c) = m(\angle g) + m(\angle b) + m(\angle c)$
Each measure in the left hand sum is a measure in the right hand sum.

6. $m(\angle g) + m(\angle b) + m(\angle c) = 180$
Two names for the same number.

10-7. Parallelograms

Concepts to be developed:

1. The shortest segment from a point A to a line r lies on the line through point A perpendicular to line r. The length of this segment is called the distance from point A to line r.
2. If ℓ_1 and ℓ_2 are parallel lines, then the distance from a point on ℓ_1 to ℓ_2 is the same as the distance from any other point on ℓ_1 to ℓ_2.
3. In a plane a line perpendicular to one of two parallel lines is perpendicular to the other also.
4. The distance between two parallel lines is the length of any segment of a line perpendicular to both given lines and having an endpoint on each.
5. A polygon is a simple closed curve which is the union of line segments.
6. A quadrilateral is a polygon with four sides. A pentagon is a polygon with five sides.
7. Opposite sides of a quadrilateral are those which have no point in common. (They do not intersect.)
8. A parallelogram is a quadrilateral whose opposite sides are parallel.
9. The opposite sides of a parallelogram are congruent.
10. The interior of a parallelogram may be separated by a diagonal into two triangular regions which are equal in area and have the same shape.

Suggestions.

It is intended that as much as possible of the development be done in class. Note that the class material actually includes Problems 1 and 2 of the Exercises 10-7.

Because in developing Property 5 the drawing of parallelograms by the class would take up a large amount of time, it would be advisable to have prepared an ample supply of parallelogram cut-outs cut on the paper cutter. It would be possible to use these same cut-outs for Problem 5 of the exercises.

Answers to Questions in Text 10-7

$m(\angle TDA) = 140$, $m(\angle TCA) = 90$, $m(\angle TBA) = 30$.

$\overline{AC}$ is the shortest.

No segments shorter than $\overline{AC}$.

The shortest segment from a point A to a point of a line r is the segment from A which is perpendicular to r.

Yes. All are right angles because of Properties 2, 2a, 3 and 4.

Yes.

The intersection of opposite sides of a quadrilateral is the empty set.

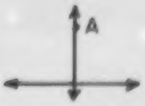
$\overline{AB}$ opposite $\overline{DC}$, $\overline{AD}$ opposite $\overline{BC}$ for both quadrilateral and parallelogram.

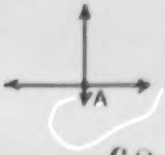
The opposite sides are congruent. Yes, Yes.

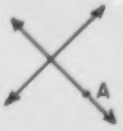
Yes, the area of the region bounded by one of the triangles is equal to one-half the area of the region bounded by the parallelogram because the triangular region is one of two equal parts of the region bounded by the parallelogram.

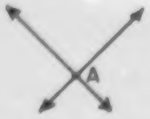
Answers to Exercises 10-7

1. As in classroom
2. As in classroom
3. (a) (d)
4.









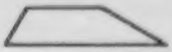
5. The triangular pieces are equal in area. A diagonal separates the region of a parallelogram into two triangular regions which are equal in area and have the same shape. (congruent)

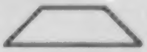
6. Yes, in a plane, lines perpendicular to one of two parallel lines are perpendicular to the other.

7. (a) $m(\overline{LM}) = 6$, $m(\overline{OM}) = 3$

   (b) $m(\overline{KL}) = 12$, $m(\overline{OK}) = 4$

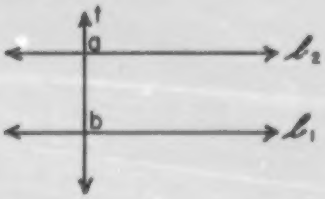
   (c) One pair of opposite sides may be equal in length. The remaining pair must be unequal in length.





*8. (a) If, in a plane, a line is perpendicular to one of several parallel lines, then it is perpendicular to the others.

   (b)



Let ℓ_2 be any line parallel to ℓ_1.

1. t is perpendicular to ℓ_2 and intersects ℓ_1	1. As drawn
2. $m(\angle a) = 90$	2. Perpendiculars form right angles
3. ℓ_1 parallel ℓ_2	3. As drawn
4. $m(\angle a) = m(\angle b)$	4. Property 2a
5. $m(\angle b) = 90$	5. Two names for the same number
6. Therefore, t is perpendicular to ℓ_1	6. If two lines intersect to form a right angle, they are perpendicular.

*9. (a) AJGF contains AJID, AJHE, DIHE, DIGF, and EHGF. BMEK contains BLIK and LMEI. IGCE is the remaining parallelogram in this listing.

(b) $\overline{DI}$, $\overline{EH}$, $\overline{FG}$.

(c) $\overline{LI}$, $\overline{ME}$.

(d) $\overline{DF}$, $\overline{EC}$.

10-8. Areas of Parallelograms and Triangles

Concepts to be developed:

1. Angles of a parallelogram at opposite vertices are congruent.
2. Consecutive angles of a parallelogram are supplementary angles.
3. If one angle of a parallelogram is a right angle, the others must be also.
4. A rectangle is a parallelogram with one right angle.
5. The area of a parallelogram may be determined, given the length of a base and the length of the altitude to that base.
6. Any side of a parallelogram may be considered its base.
7. The area of a triangle may be considered as half of the area of a parallelogram whose base is the base of the triangle and whose altitude is the altitude of the triangle to that base.
8. Any side of a triangle may be considered as a base, and for each base there is a corresponding altitude.

Suggestions

Chalkboard drawings of triangles and parallelograms should be so made that rarely do the altitudes discussed lie on a vertical line. Every effort should be made to have pupils realize that the altitudes of figures do not always extend in a vertical direction. It will also be evident that bases do not necessarily lie on horizontal lines. Closely related to this is the fact that any one of three altitudes and their corresponding bases may be used to find the area of a scalene triangle and that there are, except for a rhombus or a square, two ways to calculate the area of a parallelogram.

Consult the answer section for comments on the development of exercises in the text. Particularly note that a parallelogram with one right angle is a rectangle. If time permits, this might be a good jumping off place for a discussion of what is "necessary and sufficient" in a definition and perhaps why in some statements the "if and only if" phraseology is used.

Answers to Questions in Section 10-8

The opposite sides have the same measure.

$(\overline{AB}, \overline{DC})$; $(\overline{AD}, \overline{BC})$

$\angle ABC$, $\angle BCD$, $\angle CDA$.

There are four.

There are four transversals.

$m(\angle A) + m(\angle B) = 180$

The angles of a parallelogram at two consecutive vertices are supplementary.

The angles of a parallelogram at two opposite vertices are congruent.

If $\angle A$ is a right angle, all of the angles of the parallelogram are right angles. Yes, it is a rectangle. If $\angle A$ and $\angle B$ are not right angles and $\angle B$ is an acute angle, then $\angle A$ must be obtuse since $m(\angle A) + m(\angle B) = 180$.

Q' lies on $\overline{AB}$ extended because, as our figure shows, $\angle A$, which is an angle of ΔAQD, is congruent to the angle at B determined by $\overrightarrow{BC}$ and extension of $\overline{AB}$.

D Q Q' C is a parallelogram. $\angle D$ and $\angle DQQ'$ are consecutive angles and, therefore, supplementary. Since $m(\angle DQQ') = 90$, then $m(\angle D) = 90$.

$m(\overline{AB}) = m(\overline{QQ'})$ because each represents the sum of measures respectively equal.

The number of square units of area of a parallelogram is the product of the number of linear units in the base and the number of linear units in the altitude to the base.

The number of square units in the area of a triangle is one-half the product of the number of linear units in the base and the number of linear units in the altitude to this base.

Answers to Exercises 10-8

1. (a) 12 sq. ft. (d) 72 sq. ft.
   (b) 6 sq. yd. (e) 105 sq. m.
   (c) 80 sq. cm.
2. (a) 10 sq. in. (d) 136 sq. in.
   (b) 32 sq. cm. (e) 65 sq. ft.
   (c) 28 sq. yd
3. 30 sq. yd.
4. (a) 15,000 sq. ft.
   (b) 3,750 sq. ft.
   (c) 11,250 sq. ft.
5. $A = bh$
6. $A = \frac{1}{2} bh$
7. (a) $AB \approx 5''$ $DS \approx 2''$
   Area of $ABCD \approx 10$ sq. in.
   (b) $AD \approx 2\frac{1}{2}''$ $RB \approx 4''$
   Area of $ABCD \approx 10$ sq. in.
   (c) Yes, they do agree.

8. (a) No. It may be a trapezoid or a figure having no opposite sides parallel.

(b) No. By definition, a rectangle is a special kind of parallelogram whose angles all have a measure of 90.

(c) Yes. A square has two pairs of parallel sides. This is all that is needed to define a parallelogram.

(d) Yes. Because parallelograms, rectangles, and squares are all four-sided (quadrilateral) figures.

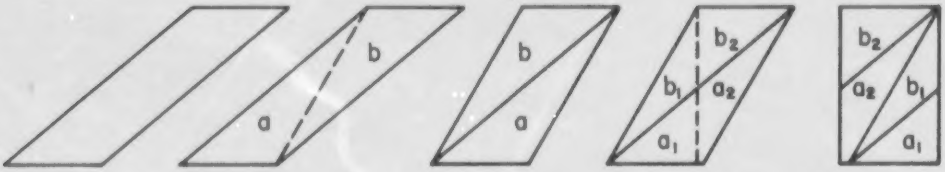
9. (a) $AB \approx 6\frac{1}{2}"$ $CD \approx 1\frac{1}{2}"$ $A \approx 4\frac{7}{8}$ sq. in.

(b) $CB \approx 4\frac{3}{4}"$ $AF \approx 2"$ $A \approx 4\frac{3}{4}$ sq. in.

(c) $AC \approx 2\frac{1}{2}"$ $BF \approx 3\frac{3}{4}"$ $A \approx 4\frac{11}{16}$ sq. in.

(d) They do agree, but are not the same.

10.



11. (a) $b + x$

(b) h

(c) b

(d) x

(e) $A = (b + x)h$, and by the distributive property, $A = bh + xh$

(f) $A = \frac{xh}{2}$

(g) $A = \frac{xh}{2}$

(h) Area of QRST = Area of QUSV less the area of RUS and QTV. The area of RUS and QTV $= \frac{xh}{2} + \frac{xh}{2} = xh$.
Area of QRST = (bh + xh) - xh
$A = bh$

10-9. Right Prisms

Concepts to be developed:

1. Definition of a right prism as a figure obtained from two polygons of same size and shape lying in parallel planes.
2. Meanings of terms edge, face, vertex, base, height, as applied to a right prism.
3. Method of obtaining volume of a right prism from the height and the area of the base.

Suggestions

Section 10-9 has been written from a quite intuitive point of view. In particular, no explicit discussion has been given of the concepts of lines perpendicular to planes or of perpendicular planes, although both are strictly involved in the idea of a right prism. One point should be brought out clearly by the teacher in classroom discussion. In the case of a right prism, if stood on one base, the upper base is "directly above" the lower base. In fact, if we imagine the prism in this position sliced horizontally (that is, parallel to the planes of the bases) then each such cut is directly above the lower base. Thus, the interior of the prism can be thought of as a series of layers or slabs piled vertically on each other. The fact that such a prism, modeled in wood, can be decomposed by simply cutting it into slabs of wood of the same size and shape and piling them on each other, is tacitly used in the discussion of the volume of the interior of a right prism. For there it is taken for granted that if we construct a slab made of cubes (or parts of cubes) just covering the base, then the top of this slab also just fits inside the prism. It was felt that this fact could better be emphasized by the teacher in class discussion than to try to develop it in detail in the student's manual.

Answers to Exercises 10-9

1. (a) 64 cubic feet
   (b) 18 cubic cm.
   (c) 30 cubic in.
2. (a) 30 cubic feet
   (b) 640 cubic meters
   (c) 126 cubic in.
3. 135 square feet for each column.
4. (a) $85\frac{1}{3}$ square feet.
   (b) $57\frac{1}{3}$ square feet.
   (c) $37\frac{1}{3}$ cubic feet.
5. $V = Bh$
6. 21 square inches. Shape of base unknown.
7. 180 cubic inches.
8. $1\frac{1}{2}$ cubic feet.
10.

Type of Prism	Edges	Faces	Vertices
Triangular	9	5	6
Pentagonal	15	7	10
Hexagonal	18	8	12
Octagonal	24	10	16

Sample Questions For Chapter 10

This is not a chapter test. Teachers should carefully select items from the following and prepare items of their own in making up a chapter test.

F 1. If one angle of an isosceles triangle has a measurement of 66°, one of the other two angles must have a measurement of 66°.

T 2. A statement may be true while its converse is false.

T 3. Pairs of corresponding congruent angles are formed when a transversal intersects two parallel lines.

T 4. A statement and its converse may both be true.

F 5. The intersection set of three lines in a plane must be three points.

T 6. If a triangle has two sides which are congruent, then it has two angles which are congruent.

F 7. All isosceles triangles have the same shape regardless of size.

T 8. The sum of the degree measures of the three angles of a triangle is equal to 180.

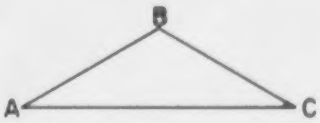
F 9. An equilateral triangle is also a scalene triangle.

F 10. The converse of a false statement is always false.

F 11. If a triangle has only two sides congruent, it can have three angles congruent.

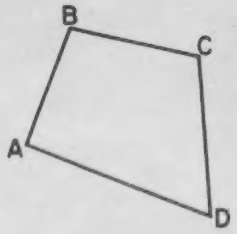
T 12. An equilateral triangle is also an isosceles triangle.

T 13. In the figure at the right, A, B, and C are symbols for the vertices of the triangle.



T 14. All parallelograms are quadrilaterals.

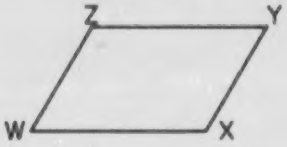
F 15. In the four-sided figure at the right, if m($\overline{AB}$) = m($\overline{CD}$) and if $\overline{AD}$ and $\overline{BC}$ are parallel, then the figure is a parallelogram.



T 16. A triangle may never include two angles whose measures are 90.

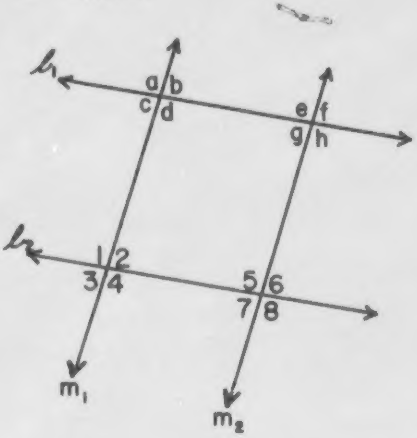
F 17. If four sides of a parallelogram are congruent, then the figure is always a square.

F 18. In the figure at the right, if $\overline{WX}$ and $\overline{YZ}$ are parallel, then WXYZ is a parallelogram.



F 19. It is possible to draw a triangle whose sides measure 4 inches, 2 inches, and 1 inch.

The figure shown at the right consists of two pairs of parallel lines. Use the figure in marking the two following statements true or false.

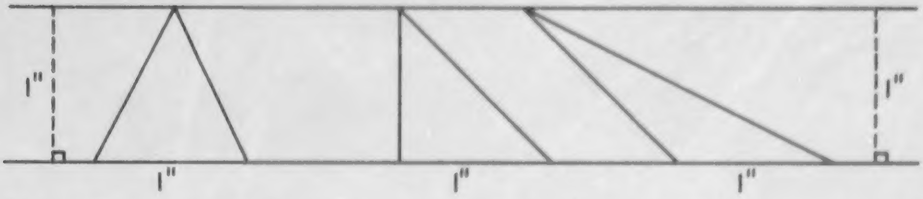


T 20. m($\angle$ a) = m($\angle$ 8)

F 21. m($\angle$ 3) = m($\angle$ h)

T 22. If m(∠ 7) = m(∠ 8) then all the angles shown in the figure are congruent.

T 23. The triangles shown below all have the same area.

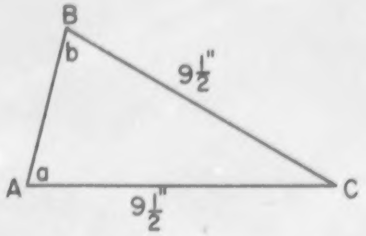


F 24. If one side of an isosceles triangle has a measure of 8, one of the other two sides must have a measure of 8.

T 25. Corresponding angles have interiors on the same side of the transversal.

F 26. If one of a pair of vertical angles measures 50, the other one of the same pair would measure 130.

T 27. In the drawing at the right, if the measure of angle A is 80, then the measure of angle B is 80.



T 28. When a line intersects two other lines in distinct points, it is called a transversal of those lines.

Multiple Choice

a 1. If in the same plane a transversal intersects two lines and the corresponding angles are congruent, then the two lines are...

(a) parallel lines.

(b) skew lines.

(c) perpendicular lines.

(d) intersecting lines.

(e) none of the above answers is correct.

c 2. If the measure of one angle of a scalene triangle is 50, which of the following statements is always true?

(a) One of the other angles has a measure of 90.

(b) One of the other angles has a measure of 50.

(c) The sum of the measures of the other two angles is 130.

(d) Two of the sides are equal.

(e) One of the other angles has a measure of 130.

b 3. In the figure shown at the right, how many transversals intersect lines m and n?

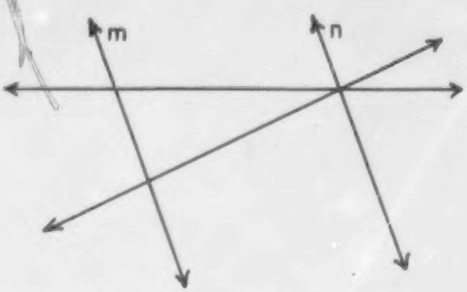
(a) 1

(b) 2

(c) 3

(d) 4

(e) 5



d 4. If the measure of one angle of a triangle is equal to the measure of another angle in the triangle then

(a) the three sides of the triangle are congruent.

(b) none of the sides are congruent.

(c) the sides opposite the angles which are congruent are not congruent.

(d) two sides are congruent.

(e) none of the above statements is correct.

e 5. If two sides of a triangle have lengths of three inches and four inches, the third side could have a measure of...

(a) one inch.

(b) seven inches.

(c) less than one inch.

(d) more than seven inches.

(e) none of the above answers is correct.

c 6. If three lines are drawn on the same plane and no two of these lines are parallel, the figure formed could include...

(a) exactly three angles.

(b) exactly two points of intersection.

(c) a triangle.

(d) two closed curves.

(e) a rectangle.

c 7. A *geometric* plane is...

(a) an airplane.

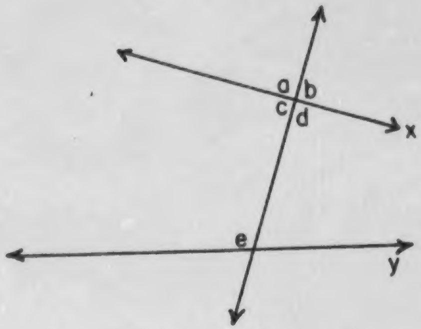
(b) a carpenter's tool.

(c) a flat surface like that of a window pane.

(d) a curved surface like that of a round lamp shade.

(e) an object like a sheet of plywood.

a 8. In the figure at the right, which angle forms with angle e a pair of corresponding angles?



(a) a

(b) b

(c) c

(d) d

(e) e

d 9. In the figure above, lines x any y are parallel *if*...

(a) angle a is congruent to angle d.

(b) angle c is congruent to angle e.

(c) angle b is congruent to angle d.

(d) angle e is congruent to angle a.

(e) angle a is congruent to angle b.

e 10. In the figure of Problem 9, if lines x and y are not necessarily parallel and if the measure of angle e is 100, the measure of angle c is...

(a) 100

(b) 80

(c) 20

(d) 10

(e) unknown

a 11. In the figure of Problem 9, an angle adjacent to angle a is ...

(a) b

(b) d

(c) e

(d) angles b, c, and d are all angles adjacent to angle a.

(e) none of the above answers is correct.

d 12. A polygon may have...

I. one side.

II. two sides.

III. three sides.

IV. more than three sides.

(a) I and II are correct.

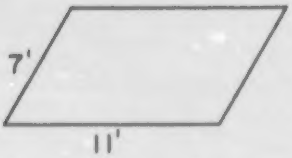
(b) II and III are correct.

(c) II, III, and IV are correct.

(d) IV is correct.

(e) none of the answers given above is correct.

e 13. The area of the parallelogram shown at the right may be found by...



(a) adding 11 and 7.

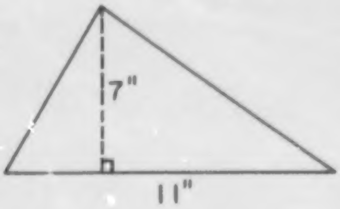
(b) multiplying 11 and 7.

(c) multiplying 11 and 7 and dividing the product by two.

(d) multiplying (11 + 7) by $\frac{1}{2}$.

(e) none of the above answers is correct.

e 14. The area of the triangle shown at the right may be found by...



I. $\frac{11 \times 7}{2}$

II. $\frac{1}{2} \times 11 \times 7$

III. $\frac{1}{2}(11 \times 7)$

IV. $7 \times \frac{1}{2} \times 11$

(a) only I and III are correct.

(b) only II and IV are correct.

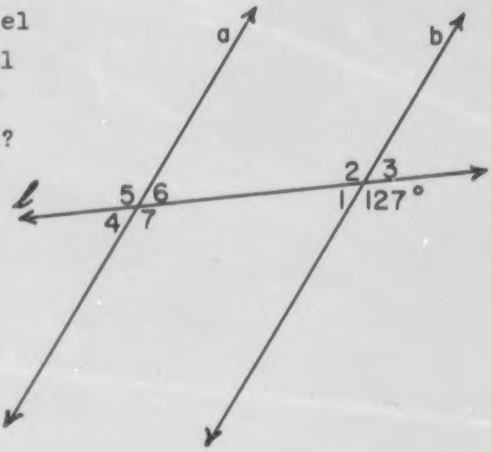
(c) only I and II are correct.

(d) only I, II, and III are correct.

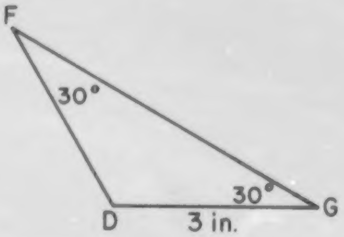
(e) all of the above are correct.

Completion

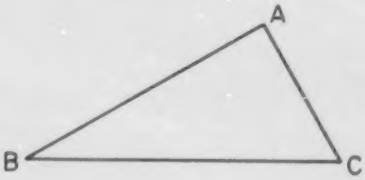
In the figure at the right lines a and b are parallel and intersect the transversal ℓ. What are the measures of each of the following angles? Note: the measurement of one of the angles is given in the figure.



1. angle 6 (53°)
2. angle 2 (127°)
3. angle 1 (53°)
4. angle 5 (127°)
5. In the figure at the right the length of segment $\overline{DF}$ is (3 in.) .

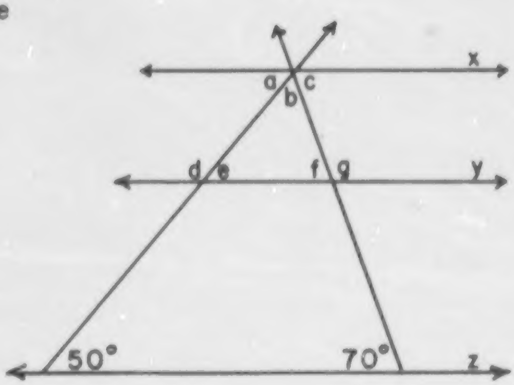


6. One of a pair of vertical angles measures 40; the other angle of the same pair would measure (40) .
7. Two lines intersect at point A. If the measure of one angle formed is 70, an adjacent angle has a measure of (110) .
8. In the triangle at the right, m(∠ ABC) = 30, m(∠ BCA) = 70 . What is the measure of angle CAB? (80)



9. Corresponding angles have interiors on (the same) side(s) of the transversal.

In the figure at the right, lines x, y, and z are parallel. What is the measure of each of the following angles?



10. angle a __(50)__

11. angle b __(60)__

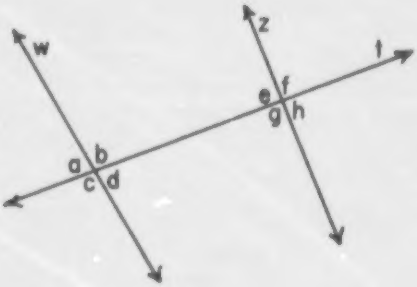
12. angle c __(70)__

13. angle d __(130)__

14. angle e __(50)__

15. When a line intersects two other lines in distinct points, it is called a (an) __(transversal)__ of those lines.

Using the figure at the right, predict whether lines w and z will be parallel or will intersect. If they intersect, indicate on which side of t (above t or below t) the intersection will occur. Fill in the space which correctly completes the following statements. (Do not write on this paper. Indicate answers on answer sheet.)

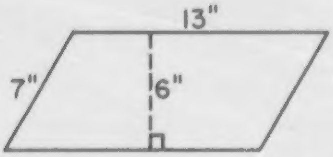


	intersect above t	intersect below t	be parallel
17. If m(∠ a) = 75 and m(∠ e) = 75, then the lines will	( )	( )	(x)
18. If m(∠ b) = 100 and m(∠ e) = 80, then the lines will	( )	( )	(x)

	intersect above t	intersect below t	be parallel
19. If $m(\angle c) = 120$ and $m(\angle g) = 100$ then the lines will	( )	(x)	( )
20. If $m(\angle d) = 60$ and $m(\angle e) = 80$, then the lines will	( )	(x)	( )

Find the areas of the following:

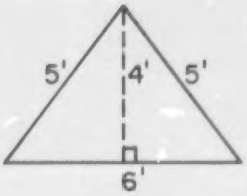
21.



PARALLELOGRAM

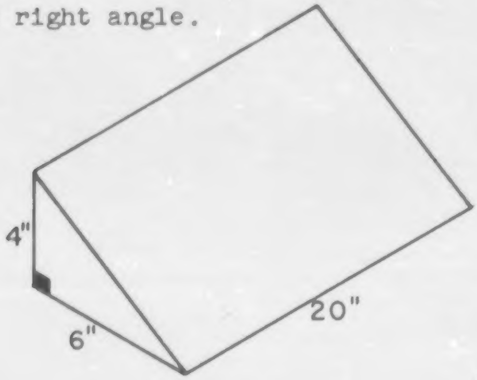
(78 sq. in.)

22.



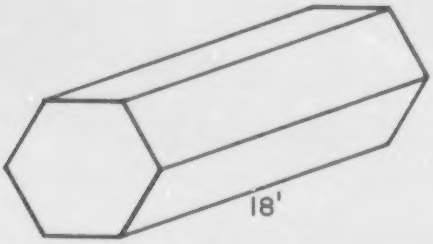
(12 sq. ft.)

23. Find the volume of the following figure. Note the indicated right angle.



(240 cu. in.)

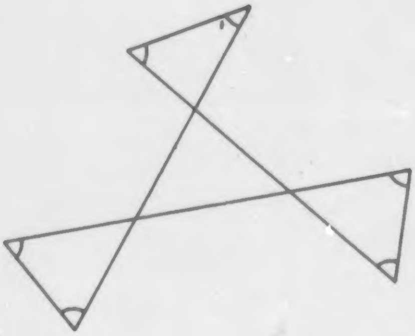
24. The area of the hexagon is $2\frac{1}{2}$ square feet. Find the volume of the prism.



(45 cu. ft.)

Very Difficult Questions-Multiple Choice.

c 1. The plane figure on the right is composed of intersecting straight lines. What is the sum of the measures of the marked angles?



(a) 180

(b) 270

(c) 360

(d) 540

(e) It cannot be determined.

c 2. In the figure below PQRS is a rectangle. TPQ and MQR are triangles. Let n represent the measure of the area TPQ and y that of MRQ. If $ST = \frac{1}{3}SR$ and $SM = \frac{1}{4}SP$, then $\frac{n}{y} = (?)$ (ST means the measure of the length of segment $\overline{ST}$.)

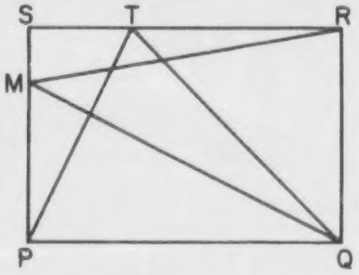
(a) $\frac{3}{4}$

(b) $\frac{4}{3}$

(c) 1

(d) 3

(e) 4



c 3. If the measure of the vertex angle of an isosceles triangle varies from 40 to 70, then the measure of one base angle varies from

(a) 55 to 70

(b) 70 to 40

(c) 70 to 55

(d) 70 to 110

(e) 140 to 110

Chapter 11

CIRCLES

The main purposes of this chapter are:

1. to acquaint the pupils with the circle, its length and area, and with some of its elementary applications to cylindrical solids;
2. to develop precision of expression and thought, and
3. to develop geometric awareness and intuition including understanding of appropriate methods of study.

Except for the study of the circle itself, the general purposes of this chapter are like those of Chapters 4 and 9. The teacher may choose to read again the introduction to the commentaries for these chapters.

An effort has been made to use accurate statements concerning the distinction between length and measure of a length. This is done in the spirit of Chapter 7. The word "radius" has two meanings as explained in the text. Also, the number r is not the radius but rather the measure of the radius.

Because in ordinary speech distinctions have not been made between length and measure of a length, many teachers may find themselves interchanging these terms. An important distinction is that we multiply numbers but not lengths. For many purposes this idea is not very important for seventh grade pupils. Teachers are advised to note such subtleties and to make some effort to use precise language. They should not be alarmed at their own failure to do so consistently. They should not insist on more than reasonable precision from students.

It is estimated that about 13 days will be required for this chapter, including time for testing and follow-up work on the test.

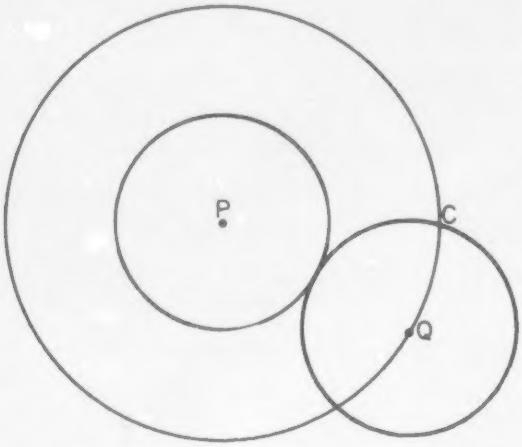
11-1. Circles and the Compass

You may wish to begin this chapter by calling attention to the prevalence of the circle in industrial and in decorative design. Some pupils enjoy bringing in pictures illustrating the use of the circle. In the history of civilization, discovery of the use of the wheel is ranked in importance with discovery of a method for producing fire.

Exercises 11-1 are planned to provide practice in following directions for drawing and labeling figures, and in use of the compass. It is important that each pupil have a compass and learn to use it correctly and with some dexterity. In following the directions for drawing figures, pupils should learn to label the figure at each stage, as subsequent directions are given in terms of the labels specified.

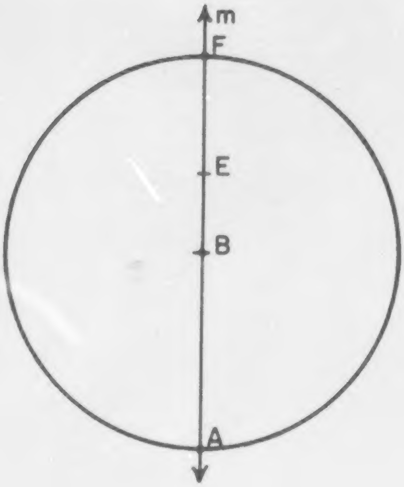
Answers to Exercises 11-1

1. (a), (b), and (c)



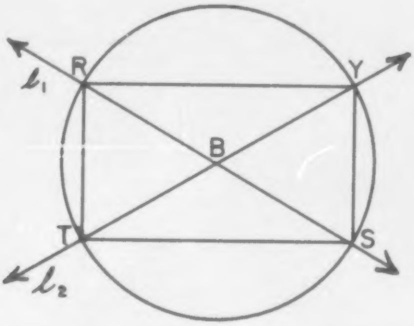
(d) Intersection is one point.

2. (a) - (e)



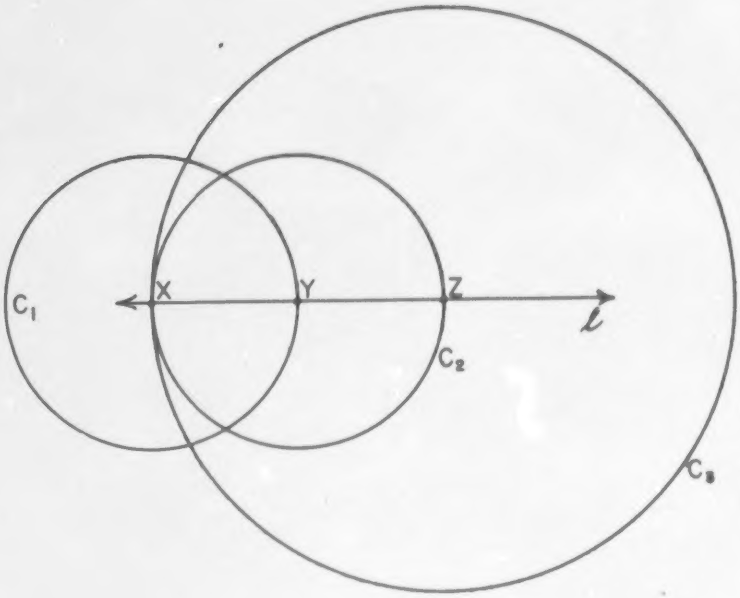
(f) Point A and Point F.

3. (a) - (c)


R
Y
ℓ_1
B
T
S
ℓ_2

(d) RTSY is a rectangle. (If the diagonals of a quadrilateral are the same length, and also bisect each other, then the quadrilateral is a rectangle.)

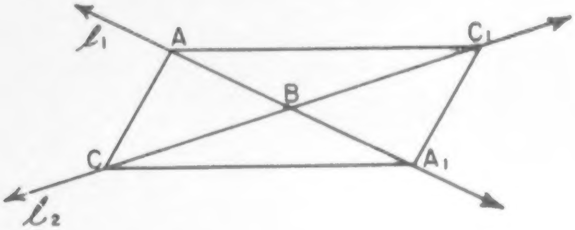
4. (a) - (d)


C₁
X
Y
Z
l
C₂
C₃

(e) Intersection set is two points.

(f) Point X

*5. (a) - (d)


l₁
A
C₁
B
C
A₁
l₂

(e) ACA_1C_1 is a parallelogram. (If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.)

11-2. Interiors and Intersections

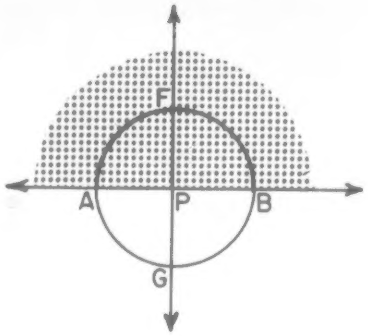
Each pupil should be repeatedly encouraged to translate written ideas into diagrams in a step-by-step manner, while reading. This cannot be overemphasized, for it applies throughout geometry.

The single point Q lies on the circle and also on $\overrightarrow{PQ}$. Two points of the circle lie on $\overrightarrow{QP}$, of which one is Q and the other needs a label. Any name is suitable; perhaps R will be suggested naturally. Intuitively R is directly opposite Q on the circle; a formal description of R is the point, other than Q, in the intersection of the circle and the line $\overleftrightarrow{PQ}$. The same is true for the intersection of the circle and $\overrightarrow{QP}$.

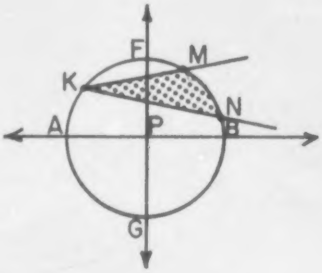
Note that $\overrightarrow{QP} \cap \overline{QR}$ is $\overline{QR}$ (if "R" was the label used as indicated above), while the intersection $\overleftrightarrow{PQ}$ and the circle consists of only the two points, Q and R.

The simple closed curve, which is the circle in this case, contains none of the points in its interior. Hence, the intersection of the interior and the circle is the empty set. Recall that in Chapter 7, the union of a simple closed curve and its interior was defined as a closed region.

The figure at the right shows the half-plane above $\overleftrightarrow{AB}$ (on the F side of $\overleftrightarrow{AB}$) and its intersection with the circle. None of the points A, G, or P belongs to the intersection, but F does. We cannot count all the points of the intersection because the intersection is an arc of the circle. An arc contains an infinite number of points.

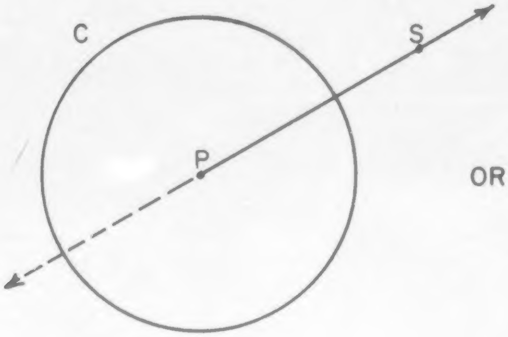


One possible choice for M, N, K is shown. The intersection of the circle and the angle consists of only the three points M, N, K. The intersection of the two interiors has been shaded in the figure.

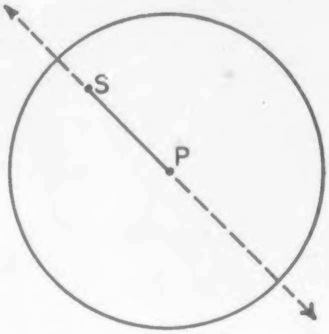


Answers to Exercises 11-2

1. (a) 1.



OR

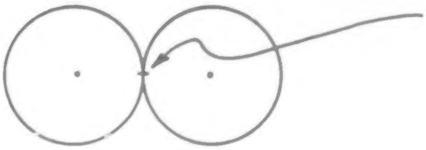


(b) 2.

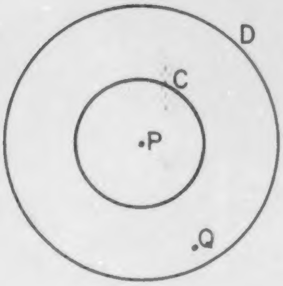
(c) No.

(d) If S lies in the interior of the circle, the number of points is 0; if S lies on the circle or outside the circle, the number is 1.

2. Yes.


intersection is one point

3.



4. (a) Same as interior of angle BPF.

(b) Intuitively, the quarter-circle in the upper right, excluding the endpoints B and F.

(c) Four of the quarters may be identified. Besides the one in (b), we have the intersection of the circle and the interior of angle APF, in the upper left; the intersection of the circle and the interior of $\angle$ APG; the intersection of the circle and the interior of $\angle$ BPG. In each case, a quarter can also be described as the intersection of the circle and two half-planes; for example, the lower left quarter is the intersection of the simple closed curve AFBG, the A-side of $\overleftrightarrow{FG}$, and the G-side of $\overleftrightarrow{AB}$.

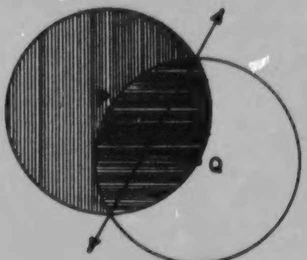
(d) Four portions of the circle might be called halves. The upper half, for example, is the intersection of the circle and the half-plane H. In Section 4, we learn that this intersection, together with its endpoints A and B, is a semicircle. Another is the lower half. the intersection of the circle and the G-side of $\overleftrightarrow{AB}$. Another is the intersection of the circle and J. The fourth one shown is the intersection of the circle and the A-side of $\overleftrightarrow{FG}$. In each of the four cases, the curve together with its endpoints is a semicircle. The pupils may wish to include the endpoints when they describe a "half" of the circle. This problem helps blaze the trail for certain notions in Section 4.

5. (a) Consists of two points.

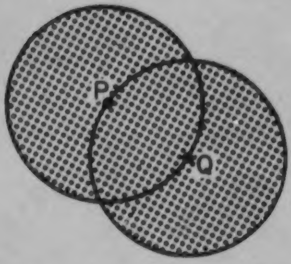
(b) Yes; no; two points determine just one line.

(c) Horizontally shaded in the figure.

(d) Vertically shaded in the figure.



(e)

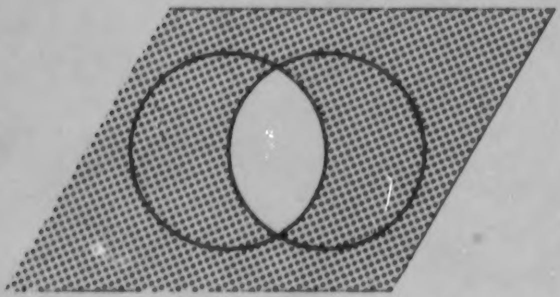


6. (a) Empty set.

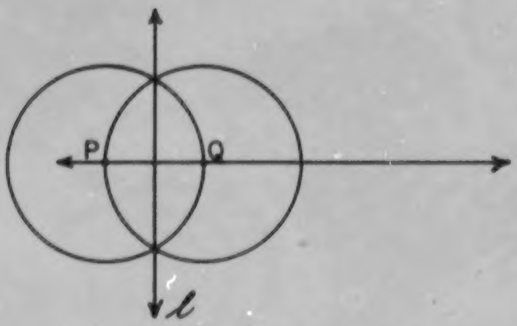
(b) Intersection of the interior of the outer circle and the exterior of the inner circle. (Note: Concentric circles are not necessarily in the same plane.)

*7. Same as exterior of the outer circle.

*8.



*9. The lines ℓ and $\overleftrightarrow{PQ}$ are perpendicular.



11-3. Diameters and Tangents

The six radii shown in the figure are $\overline{PW}$, $\overline{PA}$, $\overline{PM}$, $\overline{PV}$, $\overline{PB}$, and $\overline{PN}$.

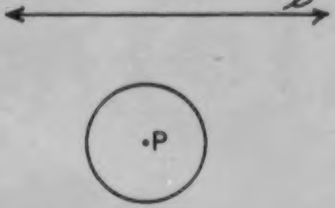
Since $d = 2r$, $r = \frac{1}{2}d$.

A circle does not need to be "on top" of a line in order for the line to be tangent to the circle.

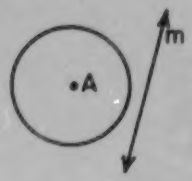
(1) The empty set. Remember that the half-plane does not contain the line.

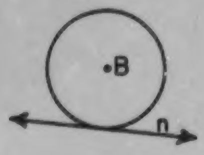
(2) The interior of the circle.

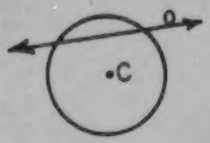
(3) Yes. See drawing at the right.



(4) No. The only possibilities are intersection sets containing 0, 1, or 2 points as shown in the drawings below







(5) (a) Four; $\overleftrightarrow{AR}$, $\overleftrightarrow{AS}$, $\overleftrightarrow{AT}$, and $\overleftrightarrow{RT}$.

(b) One; $\overleftrightarrow{AS}$, which passes through the circle.

(c) The points of tangency are E, S, and F.

(6) (a) One.

(b) One.

Answers to Exercises 11-3

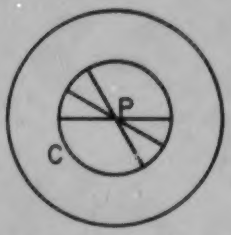
1. (a) 3 (b) 4 (c) 5

2. Answers will vary.

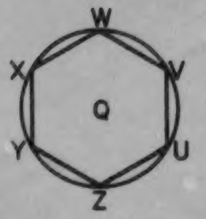
3. (a) 21 cm. (d) 2 yd.
(b) 14 in. (e) 15 ft.
(c) 5 ft.

4. (a) 12 in. (d) 10 ft.
(b) 6 m. (e) 7 ft.
(c) 34 cm.

5.



6. (a) - (f).



(Note to teachers: The radius of the circle is congruent to the chord where the endpoints of the chord mark an arc which is one-sixth of a circle.)

(g) Should be same (but the drawing is difficult to draw with the required accuracy).

(h) A diameter of the circle.

7. (a) Consists of four points R, S, T, U.

(b) Consists of Point T.

(c) Point of tangency.

(d) 4: $\overleftrightarrow{HG}$, $\overleftrightarrow{HE}$, $\overleftrightarrow{EF}$, $\overleftrightarrow{FG}$

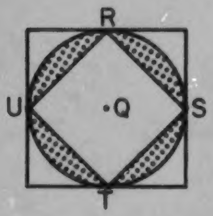
(e) R, S, T, U.

8. (a) O.

(b) Empty set.

(c) Intuitively it is composed of the four corner regions. An attempt on the pupil's part to give a careful description should convince him that the very best description is the one quoted in the problem: "the intersection of the exterior of the circle and the interior of the square EFGH." A purpose of this problem is to point out the advantages of our vocabulary in enabling us to say just what we mean with the greatest ease.

(d) As in the preceding part, the best word description is given in the problem itself. The intersection is shaded in the figure.



*9. (a) Yes. The pupil's answer may be based on the notion of symmetry, but there are many ways in which he can express his thoughts.

(b) Yes.

(c) Right angle.

(d) Perpendicular.

*10. (a) Lines are perpendicular.

(b) One.

*11. Each diameter is the union of two different radii on one line. If the intersection of two segments on one line consists of a common endpoint, then the length of their union is the sum of the lengths of the two segments. Since all radii have the same length, the measure of a diameter is twice the measure of a radius, that is, twice the measure of the radius. Since this applies to all diameters, all of them have the same length.

11-4. Arcs

In this section, the properties of an arc and a line segment are contrasted with those of a simple closed curve. Care should be taken to insure that pupils have a clear understanding of the ideas of separation and betweenness. Most pupils do not find these ideas too difficult when they think of following "paths" along segments or arcs or around a simple closed curve.

[pages 480, 481]

The symbolism for arcs is introduced in this section. Pupils should be careful to identify arcs with three points if there is a possibility of confusion between a major arc and a minor arc. These latter terms are not necessary in the development of this chapter and were not introduced in the student material. With the wealth of new terms needed in the development of the material in this chapter the pupils should not be burdened with unnecessary vocabulary.

Answers to Exercises 11-4a.

1. (a) $\overset{\frown}{XAY}$
   (b) $\overset{\frown}{YBW}$
   (c) $\overset{\frown}{WCR}$
   (d) $\overset{\frown}{CRX}$
   (e) $\overset{\frown}{AXR}$
   (f) $\overset{\frown}{AYB}$
   (Note: $\overset{\frown}{XAY}$, is the same as $\overset{\frown}{YAX}$, etc.).
2. (a) W
   (b) Y, B, and W
   (c) B, Y, and A
   (d) R, X, A, and Y
   (e) R, C, W, and B
   (f) Two answers are possible, X and R or Y, B, and W.
3. (a) A, B
   (b) A, R
   (c) A, B
   (d) Y, C
   (e) W, A
   (f) B, C
4. No. Because in naming an arc, the first and last letter designate the endpoints.

5. (a) No. Point H is between C and D, but in the other direction from C we find points K and L between C and D.

   (b) Yes. Point H.

   (c) $\overset{\frown}{CKL}$ and $\overset{\frown}{LDH}$.

   (d) No. One point does not separate a circle into two parts. Moving clockwise or counter-clockwise from L we would always come back to L.

6. (a) Yes

   (b) Yes

   (c) Yes, the endpoints

   (d) More like those of a line.

7. (a) None (c) A, B

   (b) None (d) None

8. (a) Yes

   (b) Yes

   (c) Yes

   (d) Yes

   For each ray from P passing through a point of $\overset{\frown}{AMB}$ there is a corresponding point on $\overline{FG}$. (For each point on $\overset{\frown}{AMB}$ there is a ray from P which passes through the point and also passes through a corresponding point of $\overline{FG}$.)

11-5. Central Angles

The definitions and qualifications for central angle and arc-degree are especially important in this section. Care should be taken to insure that discussions of angle are consistent with those in Chapter 7.

"Straight angle" may be a troublesome notion. A "straight angle" or "central angle of 180°" does not fit our description of "angle." An angle has been defined as a set of points that consists of two rays which have the same endpoint (the union of two rays), but the rays do not lie in the same line. This precise definition accounts for the fact that in the text, the term "special" is applied to the mention of the "straight angle."

There is also the difficulty of dealing with the so-called "reflex" angles, or angles having a degree measure greater than 180. To think of the measurement of such angles in terms of our presentation in Chapter 7 creates difficulty in differentiating between the "interior" and the "exterior" of an angle. At times however, it may be necessary to think of such angles, as in the making of circle (or "pie graphs"). In such cases, the use of the term "arc-degree" (or "degrees of arc") should eliminate some of the difficulties that may arise.

Problem 7 in the exercises is intended to introduce the pupils to the measurement of the circumference of a circle. This is done to prepare pupils for th[illegible]t section in the chapter dealing with circumferences.

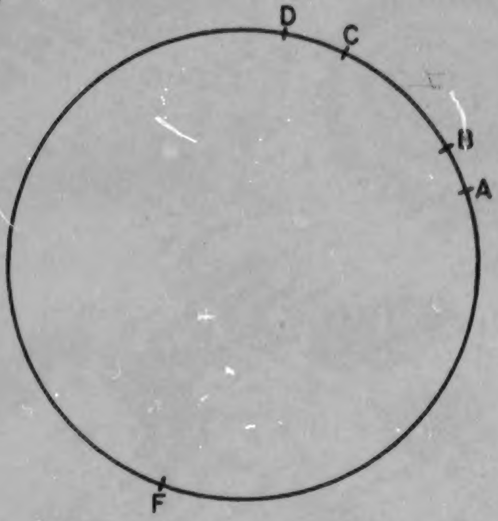
Answers to Exercises 11-5

1. (a) $m(\widehat{ABC}) = 118$ (d) $m(\widehat{BCD}) = 121$

   (b) $m(\widehat{ABCD}) = 136$ (e) $m(\widehat{CDE}) = 101$

   (c) $m(\widehat{DE}) = 83$

2. (a) - (d)



(e) 35°

3. Quarter of a circle: 90°
One-eighth of a circle: 45°
One-sixth of a circle: 60°
Three-fourths of a circle: 270°

4. (a) Yes
(b) 6
(c) 60

5. (a) $\overset{\frown}{BC}$
(b) $\overset{\frown}{DF}$
(c) $\overset{\frown}{CD}$
(d) D (the Point D)
(e) $\overset{\frown}{DE}$

6. (a) $\overline{EF}$ (b) $\overline{AB}$ (c) C (d) C

(e) It is difficult to measure along a curved line with an instrument designed to measure along a straight line. Most instruments used in measuring linear units are primarily for straight line measurement.

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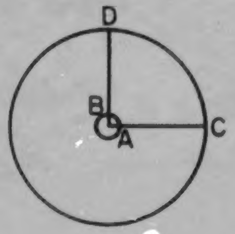
7. (a) through (c) Results will vary

   (d) No. They tend to increase, but there does not seem to be a definite pattern.

   (e) Yes. The results are all about the same; about 3.

8. In the drawing at the right, $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ each have 90 arc degrees, but the length of $\overset{\frown}{CD}$ is much greater than the length of $\overset{\frown}{AB}$.



9. Sketch a circle and sketch a diameter. Drawing in a few lines through the center which intersect both semicircles should show that a one-to-one correspondence can be established between the set of points on each of the two semicircles. This method is not unique. Another method involves perpendiculars drawn to the diameter. These set up another one-to-one correspondence between the sets of points of the two semicircles. Originality in ways of looking at things should be encouraged.

11-6. Length of a Circle

The relation between the length of a circle and the length of its diameter is introduced intuitively by having the pupils measure circles and their diameters in Problem 7, Exercises 11-5. Pupils should understand that the difference between the two measures, "c - d", is a method of comparison, but one which does not show a relation between c and d.

Pupils will have varying results with their measures, but the majority should discover that the circumference is a little more than three times as long as the diameter. This relation is the important part of this section, and the pupil must first

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understand this relation before he can appreciate the value of π to the nearest hundredth, etc. That this relation is a constant for all circles should be more apparent to the pupils if they find that other members of the class arrived at the same conclusion. It might be suggested that pupils report their measurements, tabulate them, and then propose relations which might exist before guiding them too soon and too firmly toward the ratio.

An alternative method for developing the relation between the circumference and the diameter of a circle is given here: (Method II.)

(a) On stiff paper or cardboard mark a point P and draw a circle with center P. Make the diameter of the circle between 2 inches and $2\frac{1}{2}$ inches. Then cut along the circle so that you have a circular figure with the circle you drew as its boundary.

(b) Draw a line ℓ about 10 inches long, and label as A a point near the left end of the line. Then locate point B on ℓ so that the segment $\overline{AB}$ has the same length as the diameter of your circle.

(c) Lay out a number scale on line ℓ, with 0 at point A and 1 at point B. Extend the number scale to the number 4 or 5.

(d) Now mark a point on your circle and name it C. Place the circle so it is tangent to your number line, with C at the zero point.

(e) Carefully roll the circle to the right along the number line. Each point on the circle will touch a point on the line. Continue rolling the circle until point C is again on the line. Mark as D the point where C touches the line.

(f) Between what whole numbers is point D? Estimate, to the nearest tenth, the decimal number which corresponds to point D.

(g) When you rolled circle P along the number line, each point on the circle touched exactly one point of the line, and no two points of the circle touched the same point of the line. What segment on the number line has the same length as the circle?

(h) What segment on the number line has the same length as the diameter of the circle?

(i) How does the length of the circle seem to compare with the length of its diameter?

(j) Does this result agree with your answer to Problem 1(f)?

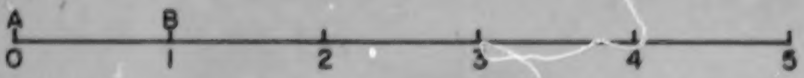
Do the results of your experimentation suggest the following statements?

1. For any circle, the ratio of the length of the circle to the length of its diameter is always the same number.
2. This number is a little greater than 3.
3. If experiments are carefully done, results suggest that this number is between 3.1 and 3.2.

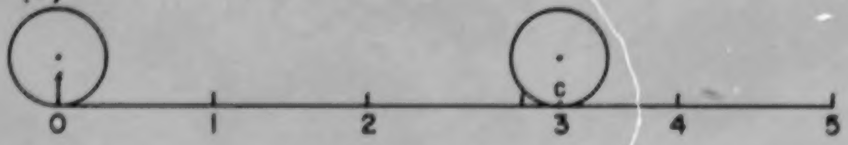
This method may be helpful in establishing the notion that π is a number, and that the symbol "π" is to be regarded as a numeral. This is the first number of the set of irrational numbers the pupils have met, and they should begin to appreciate the fact that (1) such numbers correspond to points on the number line, and (2) that a decimal expression for the number, to any desired precision, may be written. Although π is not a rational number, the assumption is tacitly made here that it behaves like an ordinary number in combining with other numbers; that is, the commutative, associative, and distributive properties for addition and multiplication hold. The optional Section 11-6b extends these ideas concerning π.

(Method II.)

(b) - (c)



(d) - (e)



(f) Between 3 and 4. Estimates will probably vary from 3.0 to 3.3.

(g) $\overline{AD}$

(h) $\overline{AB}$

(i) Length of the circle is about 3.0 - 3.3 times the length of the diameter.

(j) Results should agree fairly closely.

Answers to Exercises 11-6a

1.

	radius	diameter	(App.) circumference
(a)	———	20 in.	62.8 in. or 63 in.
(b)	7.5 ft.	———	47.1 ft. or 47 ft.
(c)	———	App. 7.2 yd.	22.6 yd. or 23 yd.
(d)	———	App. 8.4 cm.	26.376 cm. or 26 cm.
(e)	2.8 in.	———	17.584 in. or 18 in.

2. (a) π.

(b) Find the product of the diameter and π.

(c) Yes. By dividing the measure of the circumference by π.

3. (a) $\pi = \frac{c}{d}$

(b) $c = \pi d$

(c) $\frac{c}{\pi} = d$

4.

	(App.) Radius	(App.) Diameter
(a)	$3\frac{1}{2}$ ft.	7 ft.
(b)	4 in.	$7\frac{21}{22}$ in. or 8 in.
(c)	$2\frac{1}{2}$ cm.	$5\frac{1}{11}$ cm. or 5 cm.
(d)	7 yd.	$13\frac{15}{22}$ yd. or 14 yd.
(e)	14 mm.	28 mm.

5. (a) Divide the measure of the diameter by two.

(b) Multiply the measure of the radius by two.

(c) Find the product of 2, π, and the measure of the radius.

6. (a) $C = \pi(2 \cdot r)$

(b) $C = (\pi \cdot 2)r$

(c) $C = (2 \cdot \pi)r$

(d) $\frac{C}{2\pi} = r$

(e) $\frac{C}{r} = 2\pi$

(f) $\frac{C}{2r} = \pi$

7.

	Radius	Circumference
(a)	________	App. 31 in.
(b)	App. 8.2 ft. or 8 ft.	________
(c)	________	App. 105.4 cm. or 105 cm.
(d)	App. 31.9 yd. or 32 yd.	________

8. 37.68 in., or 38 in.

9. 19.7 in., or 20 in. Yes.

10. 94.2 ft., or 94 ft.

11. 3.82 ft., or 3.8 ft.

12. (a) 62.0 in.
    (b) 7.75 in., or 8 in.
    (c) 45 degrees.
    (d) .17 in., or .2 in.

(OPTIONAL) The Number π

Many pupils are interested in the fact that mathematicians have studied the properties of this number for centuries, and continue to do so; and are entertained by seeing the decimal for π to so many decimal places.

The process of describing the position of π on the number line by defining the segment on which it lies is important as an illustration of a method of analysis, and also for the illumination it sheds on the significance of successive places in decimal notation. It is recommended that this material be worked through very carefully. Teachers may choose to introduce the term "irrational." However, this is not essential for this chapter.

In the set of Exercises 11-6b, your attention is directed to Problem 3. The idea of using π as a numeral, and expressing answers in terms of π, is a useful and important one. In general, it is desirable that all answers be expressed first in terms of π, with the decimal evaluation the last step in computation when a decimal answer is required.

Pupils may be amused by a mnemonic device for remembering the first figures in the decimal for π. It is a rhyme in which the number of letters in each word indicates the digit:

> "See, I have a rhyme assisting
> My feeble brain, its task sometime resisting."

Answer to question in text: Endpoints are 3.141 and 3.142.

Answers to Exercises 11-6b

1. (a) .025, .001, .017
   (b) $\frac{22}{7}$
   (c) $\approx$ 44 in.
   (d) $\approx$ 132 ft.
   (e) $\approx$ 42 in.
   (f) $\approx$ 7 ft.
   (g) $\approx$ 66 in.
2. (a) 6.2832
   (b) 9.42477
3. (a) Diameter, 54 in. Radius, 27 in.
   (b) $(13 \cdot \pi)$ in.
   (c) $(7.2 \cdot \pi)$ in.
4. Circumference of Circle C is three times circumference of Circle D.
5. (a) Length of Circle C: 43.4 in., or 43 in.
   Length of Circle D: 31.0 in., or 31 in.
   (b) $m(ST) \approx 70$.
   $m(QR) \approx 70$.
   (c) $\frac{7}{36}$ $\frac{7}{36}$
   (d) 8.4 in., or 8 in.
   6.0 in., or 6 in.

6. (a) If the circumference is increased by 1, the measure of the new diameter is $\frac{c + 1}{\pi}$, or $\frac{c}{\pi} + \frac{1}{\pi}$. The radius has thus been increased by $\frac{1}{2}\left(\frac{1}{\pi}\right)$ ft. which amounts to about 1.9 inches. This would permit a mouse to crawl under the band.

   (b) No. If the band is made $6\frac{1}{4}$ feet longer than the original tightly fitting band, the new radius of the enlarged band will be about 1 foot greater than the radius of the original band, allowing 1 foot of crawl space.

   (c) $\frac{c}{2\pi} = r$; distributive property.

$$d = \frac{1}{\pi}\ (c + 1) = \frac{c}{\pi} + \frac{1}{\pi}.$$

11-7. Area of a Circle

Chapter 8 explains the basic method for *finding* the area of a closed region. In the case of a circle, it is preferable to have a *method* for computing the area in terms of the radius. One approach is presented in the material to be read by the pupils. Careful discussion of the questions in this section should lead most pupils to discovering intuitively that the area of a circle is a little more than three times the square of its radius. This should suggest that π has a place in computing the areas of circles.

I. (a) The area of the circle is less than the area of ABEF.

   (b) The area of the circle is greater than the area of VAZP.

   (c) The area of the circle is less than four times the area of VAZP.

   (d) No.

In counting the units of area, we may take advantage of the symmetry and count in just one quarter. In the upper right quarter, for example, 69 square regions lie entirely within the circle (by rows, from the top to the middle of the figure, 0, 4, 6, 7, 8, 8, 9, 9, 9, 9). An additional 10 square regions have the property that the larger portion is inside the circle (again, by rows, from the top, 3, 1, 1, 1, 0, 1, 0, 1, 1, 1). It is not difficult to find an approximate balance between the portions of these outside the circle and the small pieces of the remaining 7 regions which contribute to the inside. (The 7 regions, by rows, from the top, occur: 2, 1, 1, 0, 1, 1, 1, 0, 0, 0). Note that in each quarter of the figure there are drawn two square regions of which no part lies inside the circle. After counting (69 and 10), we assign the measure 79 to the area of a quarter. Thus, for the whole circle, the measure of its area is 316. Note that the value 79 is the best possible counting number that can be expected, since the theoretical measure is 78.54, to the nearest hundredth.

If children draw their own circles and measure with this counting procedure, there may be considerable variation in the results because of inadequate drawing performance and/or equipment. Most of the development should be based on the professionally drawn diagram in the book.

II. (a) The area of the circle is less than the area of BEFG.

(b) Yes. The area of the circle is a little more than three times the area of ABCD.

(c) Some pupils may recognize that the square of the radius must be multiplied by π to obtain the area of the circle. In class discussion this may logically be suggested by some pupils who have not known this before. Others may be repeating information they have gained elsewhere. It is important to the understanding of the number sentence, $A = \pi r^2$, that pupils see the value of this means of computing the area of a circle.

The area of an eight-inch square skillet is slightly less than the area of a nine-inch circular skillet. The difference would not be significant to a cook.

An alternate approach to finding the area of a circle is presented in the last problem of Exercise 11-7. A third approach is presented here, and teachers may find it useful to present this to some members of the class. Capable students should be able to follow the discussion here with little or no assistance.

Let us consider a circle and suppose that the number r is the measure of a radius. Suppose that many radii of the circle are drawn in a regular pattern. The diagram now resembles a wheel with many spokes. In Figure 11-7 we chose to have sixty radii. In order to have a definite number to speak about in the remainder of the discussion, we will continue to use this choice of 60. The sixty rays separate the circle into sixty arcs, all of the same length. They also separate the interior of the circle into sixty regions, all of the same area.

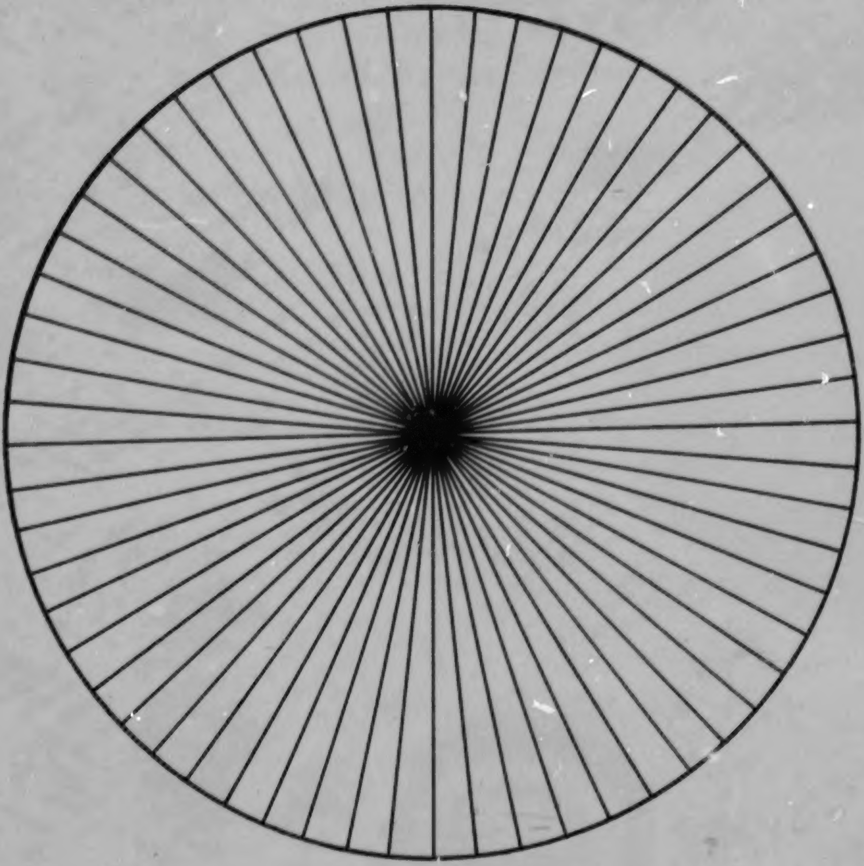
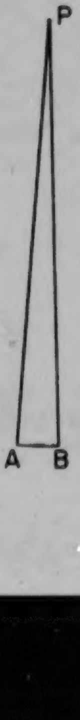


Figure 11-7



We fix our attention on a typical one of these regions, as shown in the figure. This region appears to be nearly triangular; indeed, only a short portion of its boundary, namely the arc $\widehat{AB}$, is not straight. Perhaps we can approximate the area of this region by finding the area of the interior of a triangle. (Do you recall the method for computing the area of a triangular region?) We choose the triangle whose base is as long as the arc $\widehat{AB}$ and whose altitude is the same as the distance of P from the arc.

Since the measure of the length of the entire circle is $2\pi \cdot r$, the measure of the length of the arc is $\frac{1}{60} \cdot 2\pi r$.

The distance between P and the arc is simply the radius of the circle.

Thus for the approximating triangle, as described, the measure of its area is $\frac{1}{2}\left(\frac{1}{60} \cdot 2\pi r\right)(r) = \frac{1}{60} \cdot \pi \cdot r \cdot r$.

An approximation to the measure of the area of the region bounded by the curve ABP is $\frac{1}{60} \cdot \pi r^2$. Since there are sixty such regions inside the circle (Figure 11-7), an estimate of the measure of area of the circle is 60 times $\frac{1}{60} \cdot \pi r^2$, that is, πr^2.

How does this approximation compare with the results obtained by actual measurements which you made earlier?

In reality the interior of a circle with radius r units of length has exactly πr^2 units of area. Thus, although our method of development used an approximating step, our result agrees with the correct result that may be firmly established in more advanced mathematics.

Answers to Exercises 11-7

1. (a) 154 sq. in.
   (b) $78\frac{4}{7}$ (or 79) sq. ft.
   (c) 616 sq. cm.
   (d) 1386 sq. yd.
   (e) 38.5 (or 39) sq. mm.
   (f) 55.44 (or 55) sq. yd.
2. (a) 200.96 (or 201) sq. ft.
   (b) 314 sq. yd.
   (c) 706.5 (or 707) sq. cm.
   (d) 1256 sq. ft.
   (e) 1017.4 (or 1017) sq. in.
   (f) 19.63 (or 20) sq. yd.

3.

Circle	Radius	Diameter	Circumference	Area
(a) A	______	8 ft.	24.8 (or 25) ft.	49.6 (or 50) sq. ft.
(b) B	8 cm.	______	49.6 (or 50) cm.	198.4 (or 198) sq. cm.
(c) C	10 ft.	______	62 ft.	310 sq. ft.
(d) D	16 mi.	32.2 (or 32) mi.	______	793.6 (or 794) sq. mi.
(e) E	______	222 in.	688.2 or 688 in.	38195.1 (or 38195) sq. in.

4. Since $49 < 50.24$, the circular skillet.

5. 36π square inches, or approximately 113 square inches.

6. (a) First. (b) yes.

7. $40 \cdot 30 - 7^2 \cdot \pi$; to the nearest square foot, 1046.

8. (a) 16π.
   (b) 16.
   (c) 4.
   (d) 8.
   (e) 8π
   (f) 4π.
   (g) $8 + 4\pi$.

*9. Approximately 70 quadrillion square kilometers (70,000,000,000,000,000).

*10. 3.

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*11. $22\frac{1}{2}$; parallelogram whose base is πr units, and whose altitude is r units. This activity requires considerable care in the construction and cutting of the diagram. Result should be πr^2.

12. BRAINBUSTER:

(a) Chain at corner.

(b) 785.4 square feet.

Answers to Exercises 11-8

The numerical answers obtained when computing volumes of cylinders sometimes appear more precise than the indicated accuracy of the original measurements would justify. It is reasonable to round some of the answers.

1. (a) 396.8 (or about 397) cubic inches

(b) 793.6 (or about 794) cubic feet

(c) 9300 cubic centimeters

(d) 3797.5 (or about 3798) cubic yards

(e) 5356.8 (or about 5357) cubic inches

2. (a) about 140 cubic feet

(b) about 19 cubic feet

(c) about 37 cubic inches

(d) about 4464 cubic yards

(e) about 232 cubic inches or about .13 cubic feet

3. $V \approx 3391.2$ or, Volume is 3391.2 cu. ft. (3391, 3390, or 3400 could be used).

4. Volume is 2π cu. ft., 6 cu. ft.

5. 47 gallons (or 45 gallons if one computes from 6 cu. ft.)

6. $V \approx 3768$. Or Volume is about 3770 cu. in.

*7. $V \approx 90\pi$.

*8. $V \approx 180\pi$.

*9. $V \approx 360\pi$.

*10. $V \approx 720\pi$.

*11. It doubles. (V is multiplied by two.)
It quadruples (V is multiplied by four.)
V is multiplied by eight.

*12. BRAINBUSTER.
About 52 cu. in.

Answers to Exercises 11-9.

1.

	Circle	Radius	Diameter	Height	Total Surface Area (A_t)
(a)	A	5 in.	—	—	465 (sq. in.)
(b)	B	—	2 ft.	—	24.8 or 25 (sq. ft.)
(c)	C	8 ft.	—	—	1240 (sq. ft.)
(d)	D	—	30 cm.	—	6045 (sq. cm.)
(e)	E	4 yd.	—	—	396.8 or 397 (sq. yd.)

2. (a) About 125.55 or 126 (sq. in.)

(b) About 5425 (sq. ft.)

(c) About 1841.4 or 1841 (sq. in.)
or
About 13.95 or 14 (sq. ft.)

(d) About 6780.25 or 6780 (sq. ft.)

3. HOME PROBLEM.

(a) Rectangle (or rectangular region); altitude or height; circumference.

(b) (Construction problem).

(c) Use tape to find length of base circle directly.

(d) Yes.

(The numerical answers given for Problems 4, 5, and 6 imply much more accuracy than is justified by the numbers used. It would be reasonable to round them off.)

4. Lateral surface area is 75.36 sq. centimeters.

5. Total Area is 89.49 sq. centimeters.

*6. About 10.4 sq. meters; about 8.3 sq. meters.

*7. 3.5 gallons.

Answers to Exercises 11-10

1. a, b, c.

Diagram	(a)	(b)	(c)
(1)	circle	circumference	$c = 2\pi r$
(2)	rectangle	area	$A = lw$
(3)	triangle	area	$A = \frac{1}{2}bh$
(4)	circle	area	$A = \pi r^2$
(5)	rectangular prism	volume	$V = lwh$
(6)	circle	circumference	$c = \pi d$
(7)	cylinder	lateral area	$L = 2\pi rh$
(8)	triangular prism	volume	$V = \frac{1}{2}b_t h_t h_p$

(9)	parallelogram (or rectangle)	area	$A = bh$
(10)	cylinder	volume	$V = \pi r^2 h$
(11)	rectangular prism	lateral area	$L = 2\ell h + 2wh$
(12)	cylinder	total area	$T = 2\pi rh + 2\pi r^2$

2. (a) Numerals to be circled are 2, $\frac{1}{2}$, and π wherever they occur.

(b) Symbols r, l, w, b, h, d, b_t, h_t, h_p, wherever they occur.

(c) Number of arrows below the ruler is the number of dimensions of the figure.

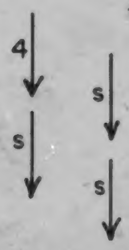
(d) See answer to Problem 2 (c). Number of measures indicates number of dimensions.

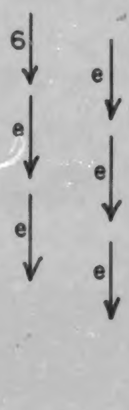
(1) 1 (3) 2 (5) 3 (7) 2 (9) 2 (11) 2
(2) 2 (4) 2 (6) 1 (8) 3 (10) 3 (12) 2

(e) Number of measures in formula which is a product indicates number of dimensions. In (11) and (12), number of measures on one branch indicates number of dimensions.

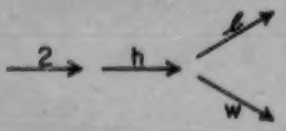
(f) "B" above bottom arrow in (5), (8), and (10).

*(g) Square: $p = 4s$ Cube: $S = 6e^2$
$A = s^2$ $V = e^3$



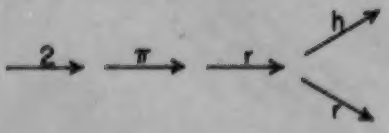


*(h) (11)



$L = 2h(\ell + w)$

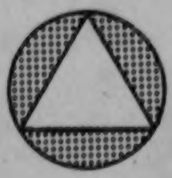
(12)



$T = 2\pi r(h + r)$

3. (a) 68°
   (b) 150°
   (c) 83°
   (d) 116°
   (e) 85°

4. (a) 519 square meters
   (b) 1257 square meters
   (c)



   (d) 738 square meters
   (e) 246 square meters

5. (a) 66 meters; 144 square meters.
   (b) 119 centimeters; 814 square centimeters.
   (c) $80 + 20\pi$ feet; $480 + 200\pi$ square feet.
   (d) $62 + \frac{3\pi}{2}$ millimeters; $84 + \frac{9\pi}{4}$ square millimeters.

6. 72.8 cu. ft.

*7. 113.6 sq. ft.

Sample Questions

Note: Teachers should construct their own tests, using carefully selected items from those given here and from items of their own. Careful attention should be given to difficulty of items and time required to complete the test.

I. True-False

T. 1. A circle is a set of points.

T. 2. A central angle of 90° is a right angle.

F 3. If a circle is divided into six equal parts, the length of each of these parts of the circle is equal to the length of the radius.

T 4. A circular waffle iron with a radius of three inches makes a larger waffle than a square waffle iron 5 inches on a side.

T 5. The line segment from a point on the circle to the center of the circle is the radius of the circle.

F 6. The exterior of a circle is the set of all points whose distance from the center is less than the length of the radius.

T 7. Two circles which have the same center are concentric.

T 8. The area of a round table top 5 feet in diameter is less than 20 square feet.

F 9. The interior of a circle is a set of points that includes the circle.

T 10. An infinite number of lines may be tangent to a circle.

T 11. The circumference of a circle is the perimeter of the circle.

T 12. An angle with vertex at the center of a circle is called a central angle.

F 13. A semicircle is half the interior of a circle.

F 14. Any line segment whose endpoints lie on the circle is the diameter of the circle.

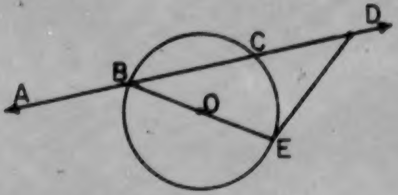
F 15. Doubling the measure of the radius of a circle will double the area of the circle.

F 16. Mathematicians have proved that 3.14 is the exact value for π .

II. Multiple-Choice

Select the choice which you consider answers the question or completes the statement for each of the following.

3 17. In the figure at the right, the intersection of line $\overleftrightarrow{AD}$ and the circle is the set of points:



(1) {A, B}

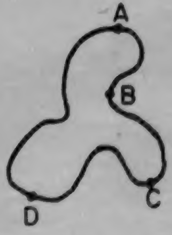
(2) {A, C}

(3) {B, C}

(4) {C, D}

(5) None of the above.

5 18. In the figure at the right, which of the following could be true?



I. D is between A and C.

II. B is between A and C.

III. D is between B and C.

(1) Only I is true.

(2) Only II is true.

(3) Only III is true.

(4) I, II are both true.

(5) I, II, and III are all true.

2 19. The intersection of a circle and one of its tangents is a set of points with:

(1) no elements

(2) one element

(3) two elements

(4) three elements

(5) an infinite number of elements

5 20. If a rope 33 inches long forms a perfect circle, the length of the diameter in inches would be about:

(1) 100

(2) 66

(3) 33

(4) 21

(5) $10\frac{1}{2}$

4 21. Which of the following are closed curves?

(1) parallelogram

(2) circle

(3) triangle

(4) (1), (2) and (3) are correct.

(5) none of these is correct.

4 22. What is the greatest number of elements in the intersection set of a circle and a triangle?

(1) zero

(2) one

(3) three

(4) six

(5) an infinite number

1 23. In the arc shown at the right what is $\widehat{BE} \cap \widehat{CF}$?

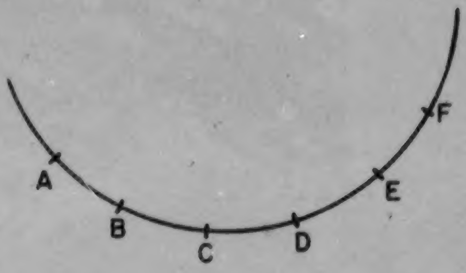
(1) $\widehat{CE}$

(2) $\widehat{DE}$

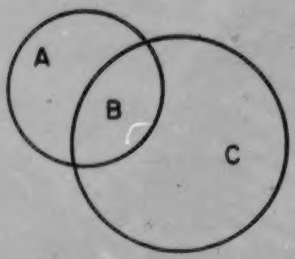
(3) $\widehat{CD}$

(4) $\widehat{BD}$

(5) $\widehat{BF}$



3 24. In the figure at the right, what is the intersection set of the interior of the longer circle and the exterior of the shorter circle?



(1) Only Area A

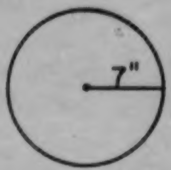
(2) Only Area B

(3) Only Area C

(4) Both Areas A and C

(5) All the area outside of A, B, and C.

4 25. If $\pi = \frac{22}{7}$ what is the measure in inches of the circumference of the circle shown at the right with a seven inch radius?



(1) 11

(2) 22

(3) 33

(4) 44

(5) 55

4 26. If $\pi = \frac{22}{7}$ the measure in square inches of the area for the circle in Question 25 is about...

(1) 122

(2) 133

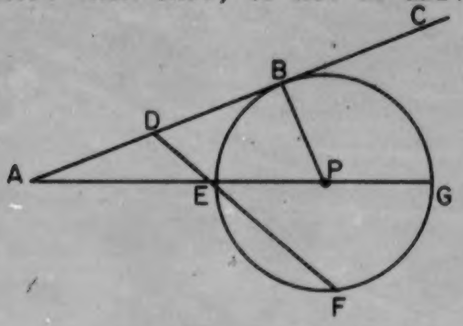
(3) 144

(4) 155

(5) 166

III. Matching

Using the drawing shown below determine what geometric terms are illustrated by the parts of the circle with center at P. Match the items in the left column with the geometric terms in the right column. The terms in the column at the right may be used once, more than once, or not at all.



5 27. $\overline{DF}$

3 28. $\overline{AC}$

1 29. $\overline{PB}$

2 30. $\overline{EG}$

4 31. $\overset{\frown}{EFG}$

5 32. $\overline{APG}$

(1) radius

(2) diameter

(3) tangent

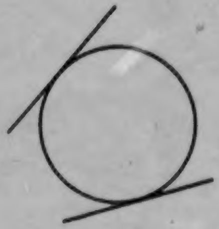
(4) semicircle

(5) none of the above

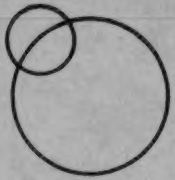
IV. Problems

Write the word or number that you think best completes each of the following statements or answers the question. (Use $\frac{22}{7}$ for π for Questions 33-37).

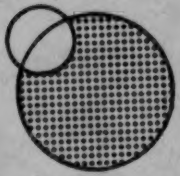
33.	A bicycle tire is 28 inches in diameter. How far does the wheel travel on the ground in one revolution?	Answer: About 88 inches
34.	Round Pizza Pies have diameters that are 14 or 21 inches. If the 14-inch Pizza costs \$1.00, what should be the price of the 21-inch Pizza at the same rate per square inch?	Answer: \$2.25
35.	What is the volume of a can of peaches that is 4 inches in diameter and 6 inches high?	Answer: $75\frac{3}{7}$ (or 75) cu. in.
36.	What is the surface area of the metal in the can for the peaches in Problem 35 above?	Answer: $100\frac{4}{7}$ (or about 100) sq. in.
37.	If a circular race track is 1 mile long, what is the radius of the track?	Answer: About $\frac{1}{6}$ mile or about $\frac{7}{44}$ miles or about 840 ft.

38.	Ellen's belt is 21 inches long and just fits around her waist. If her waist were a perfect circle, what would be its diameter?	Answer: 6.7 in.
39.	A value for π used by the ancient Egyptians is $\frac{355}{113}$. How different is this from the value we use, to two decimal places?	Answer: Same to two decimal places.
40.	A merry-go-round has three rows of horses, the outer row being 6 feet farther out than the inner row. If you sit on an outside horse, how much farther do you ride in one turn of the merry-go-round than if you sit on the inside horse?	Answer: 12π feet or 37.68 feet.
41.	The diameter of a certain circle is 60 centimeters. What is the area of this circle? (Use the decimal approximation to π to the nearest hundredth).	Answer: 2826 square centimeters.
42.	Draw a circle and two segments such that each of the segments is tangent to the circle.	One possible answer: 

43. Shade the intersection of the interior of the longer circle and the exterior of the shorter circle.

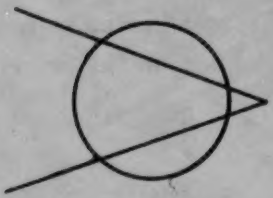


Answer:



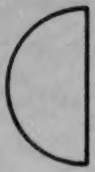
44. Draw a circle and an angle such that their intersection consists of four points.

One possible answer:



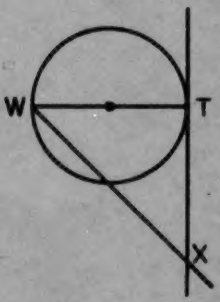
45. The simple closed curve in the figure consists of a semicircle and a segment. The radius of the circle is r units of length. Find the area of the interior of the simple closed curve. (The answer should be expressed in terms of r.)

Answer:
$\frac{1}{2}\pi r^2$
units of area



46. The segment $\overline{WT}$ is a diameter of the circle. The lines $\overleftrightarrow{TX}$ and $\overleftrightarrow{WT}$ are perpendicular. The segment $\overline{TX}$ is 3 feet long. The measurement of the angle TWX is 45 degrees.

Answer:



(a) What is the measure of $\angle$ WXT in degrees?

Answer: 45.

(b) Find the radius of the circle.

Answer: $\frac{3}{2}$ feet.

47. Find the volume of the inside of a pipe whose length is 80" and the radius of whose inside base circle is $\frac{1}{4}$". Use $\pi \approx 3.14$.

Answer: 15.70 cu. in.

48. Find the total surface area of a cylinder the radius of whose base circle is 10 centimeters if the altitude of the cylinder is 100 centimeters. Use $\pi \approx 3.1$.

Answer: 6820 square centimeters.

49. Find the amount of paper needed to make labels (without overlapping them) for 3 cans each of height 6" and of base circle radius 2". Use $\pi \approx 3.14$.

Answer: 226.08 sq. in.

50. Find the volume of a cylindrical solid the diameter of whose base circle and whose height are each 3 meters. Leave your answer in terms of π.

Answer: $\frac{27}{4}\pi$ or 6.75π cu. meters

Chapter 12

MATHEMATICAL SYSTEMS

In this chapter, it is particularly important that teachers have clearly in mind both the *objectives* of the chapter and the suggested *method of approach* to be used with it.

The *main objective* is to lead the students to achieve some *appreciation of the nature of mathematical systems*. It is neither intended nor desirable that the children memorize the various tables introduced here, or drill for mastery of the operations introduced here.

It is especially important that the teacher read this chapter through very carefully before planning his presentation, and give considerable thought to how to present some introductory motivation, and even more to how to lead the students to *discover* the various relationships and properties which appear in the chapter for themselves in advance of the reading of the text. The text itself attempts to suggest problems and processes for doing this as does this teacher's guide. However, these can be effective only if carefully planned for by the teachers. The process of discovering, of *perceiving for one's self* is a vital step in achieving our major objective: an appreciation of the nature of some types of mathematical systems. This is close to an appreciation of the nature of modern mathematics and of the work of mathematicians.

One of the most important activities of modern mathematicians is the search for common attributes or properties often found in apparently diverse situations or systems. Sometimes these common elements are deliberately built into new systems which are constructed as generalizations or abstractions of old systems, as when the number system is extended from the system of counting numbers to the whole numbers, to the rational numbers, etc., etc. Sometimes these common elements are observed in systems less clearly related at first glance, as when the changes of position of a rectangle into itself are conceived of as forming an algebraic system with a "multiplication" table, which is discussed in the chapter.

Frequently, the systems developed out of the intellectual curiosity of mathematicians and their search for patterns in diverse abstract situations have been exactly the tools needed and seized upon by scientists in their attack on the problems of our physical world. The theory of groups, which actually has as its logical beginnings the properties discussed in this chapter, had its chronological beginnings in the early 19th Century in problems relating to the solution of equations. Matrices, some of which form groups and give further examples of the principles of this chapter, were invented largely by the Englishman Cayley a little later. Within our generation the German physicist Werner Heisenberg has used matrices in the formulation of the quantum mechanics which is highly important in modern physics. Analogous stories relate the development of radio by Marconi to the differential equations of Maxwell, and point out that the outgrowths of Einstein's relativity theory owe much to his use of the tensor calculus developed by the Italian geometers Ricci and Levi-Civita. All of these stories have the same theme, namely, that both mathematicians and scientists are always seeking unifying principles or patterns. Frequently mathematics, developed solely for the intrinsic interest of its properties and structure, was later found to fit the needs of science, but for both science and mathematics we need to develop students who can see and understand patterns and structure.

In this chapter we are studying mathematical systems involving sets of elements and binary operations. Such systems which have certain simple additional properties are called groups and their study is a major branch of so-called "modern algebra." We shall not use all of these technical terms. However, other substantial objectives incidental to the major concern of this chapter and appropriate for secondary school students are:

1. Increased understanding of the nature and occurrence of the commutative, associative, and distributive properties, as well as the concepts of closure, identity element, inverse of an element.
2. Increased understanding of the inverse of an operation and its relationship to inverse and identity elements.

Additional discussions of these ideas and problem materials may be found in the books listed in the bibliography in Section 12-9.

Aside from the general considerations mentioned above, there are very specific applications of the modular systems with which this chapter is chiefly concerned. The applications to days of the week, hours of the day, days in the month are obvious and immediate. Not quite so obvious are applications to two-way switches (mod 2) which are most common, but also to n-way switches for a number of small values of n. These are used increasingly in modern computing and in industry. The recognition that all are aspects of one system - modular arithmetic - gives insight not only to mathematics but to various applications as well. This in turn is an example of periodicity -- a repetitive pattern -- that occurs so often within and outside of mathematics.

The teacher should be especially cautioned in the use of the exercises in this chapter. There are altogether too many for use in one class. To give all would lay too much stress on techniques and make a chore out of what should be an interesting development. Many exercises are given so that the teacher may use different sets in different classes and have some left over for review at the end.

The estimated time for this chapter is about 12 days.

12-1. A New Kind of Addition

Several of the sections, including the first, discuss the properties of what is referred to as modular arithmetic. The face of a clock is used to illustrate modular addition. The following quote provides background for the basic notion of this idea:

"In number theory we are often concerned with properties which are true for a whole class of integers differing from each other by multiples of a certain integer. Take, for instance, the fact that the square of an odd integer when divided by 8 leaves 1 for a remainder. Here we have a property holding for all odd numbers; that is, for a class of numbers differing from each other by multiples of 2. As another example, we see that when the last digit of a number, in decimal notation, is 6, then the last digit of its square will also be 6. Thus, in this simple example, we deal again with a property shared by integers differing by a multiple of an integer; namely, 10.

"The consideration of properties holding for all integers differing from each other by a multiple of a certain integer leads in a natural way to the notion of congruence. Two integers a and b whose difference $a - b$ is divisible by a given number m (not 0) are said to be congruent for the modulus m or simply congruent modulo m. Gauss, who introduced the notion of congruence, proposed the notation

$$a \equiv b \pmod{m}$$

to designate the congruence of a and b modulo m."[1]

Some textbooks use the following definition:

If $a = km + b$, then $a \equiv b \pmod{m}$. The $\equiv$ sign is read, "Is equivalent to" or "is congruent to."

We emphasize that there is no need for the pupil to become familiar with the terms used in the above discussion, including "modular arithmetic."

---

[1] Uspensky and Heaslet, Elementary Number Theory, McGraw-Hill, 1939, page 126.

In solving problems using replacements, encourage the pupil to make a list of possible replacements first. For mod 5 the set of possible replacements would be {0, 1, 2, 3, 4}; for mod 8, the set would be {0, 1, 2, 3, 4, 5, 6, 7}. These are examples of finite systems.

For the teacher's information each element of the set can be considered as an equivalence class, thus, numbers are put in equivalence classes.

Without a doubt, some pupils will wonder why the symbol " ≡ " ("is equivalent to") is used instead of " = ." This is an excellent opportunity to point out that the = sign is used when we have two names for the same thing; thus $3 + 2 = 4 + 1$ since these are two names for the same number, five. In the case of modular arithmetic, when we say "Five is equivalent to one, mod 4," the "five" and the "one" are not names of the same thing, thus it is necessary to introduce another symbol to describe this relationship.

Answers to Exercises 12-1

1.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(a) 0　　(c) 0

(b) 2　　(d) 1

2. mod 3

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

mod 5

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

3. (a) 0 (c) 1

(b) 1 (d) 2

The teacher may want to let the students try exercises (mod 4) before taking up other modulii.

4. (mod 6)

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(mod 7)

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

(a) 2 (c) 2

(b) 4 (d) 2

The pupils may use the tables made in Problem 3 above or make sketches of clocks.

5. $23 = 5(4) + 3$. The hand will go around four times and stop at 3.

6. Seven hours after eight o'clock is five o'clock. This is addition (mod 12).

7. Nine days after the 27th of March is the fifth of April. This is addition (mod 31) since there are 31 days in March.

12-2. A New Kind of Multiplication

This section does for multiplication what the first section did for addition. It not only gives other examples of operations for use in the next section but also prepares for modular arithmetic in a later section. The transition from getting a multiplication table by adding and to getting it by dividing and taking the remainder should be made on the initiative of the students as a means of making computation easier. It is hoped that this could be discovered by some of the students themselves. Problems 5 and 6 are designed to encourage this transition. This is certainly one place where to push a transition too rapidly can lead to trouble but where discovery in the students' own good time can be an enjoyable experience for all concerned.

Answers to Exercises 12-2

1. (a) mod 5

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

mod 6

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

mod 7

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

2. (a) 1
(b) 0
(c) 3
(d) 1
(e) 0

3. (a) 6
(b) 1
(c) 6
(d) 0

4. (a) $(7)(10) \equiv ? \pmod{31}$

$70 \equiv 8 \pmod{31}$

$4 + 8 = 12$; hence February 12 is the date 10 weeks after December 4th.

(b) $(2)(365) \equiv ? \pmod{7}$

$730 \equiv 2 \pmod{7}$

Thursday was the day of the week for August 6, 1959.

5.

mod 5

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

6. The Table is identical with the multiplication Table (mod 5). Dividing a whole number by 5 and retaining the remainder yields the same results as those obtained by subtracting the greatest multiple of five contained in a given number and retaining the remainder. It may be easier to divide and retain the remainder.

*7. (a) $x = 2$ (d) $x = 3$
(b) $x = 4$ (e) $x = 0$
(c) $x = 1$

*8. (a) impossible
(b) impossible
(c) $x = 1$, $x = 3$, $x = 5$
(d) impossible
(e) $x = 0$, $x = 2$, $x = 4$

12-3. What is an Operation?

Skills and Understandings

1. To recognize a binary operation described by a table.
2. To recognize a binary operation described in words.
3. To find, from a table, the result of putting two elements together in a binary operation described by the table.
4. To find, by computation, the result of putting two elements together in a binary operation described in words.
5. To tell, from the table for a binary operation, whether or not the operation is commutative.

6. To know that: (a) In order to show that a binary operation *is* associative, it is necessary to show that an equation [e.g. $a * (b * c) = (a * b) * c$] holds for *every* triple of elements a, b, c. (b) In order to show that a binary operation *is not* associative, it is sufficient to find *one* triple of elements a, b, c for which the equation does not hold [e.g., $a * (b * c) \neq (a * b) * c$].

Teaching Suggestions

To be given a binary operation, we must be given a set of elements and a way of combining any two elements to get a definite thing. The "definite thing" may or may not belong to the original set of elements. The two elements we combine may be the same element taken twice. If the operation is given to us by a table, the set is composed of those elements which appear in the left-hand column and in the top row (the same elements must appear in both places). For example, the set for the operation of Table (c) is {0, 1, 2, 3}; that for the operation of Table (d) is {1, 2, 3}. In Table (d), all the entries in the table belong to the set {1, 2, 3}; in Table (c) many of the entries in the table do not belong to the set {0, 1, 2, 3}. This point is discussed more fully in the next section on closure.

Bring out by class discussion that the entries in the tables (the results of putting two elements together) could be anything at all. As later examples will show, they do not have to be numbers.

Practice reading the tables. Stress that, in evaluating $1 \square 3$, the "1" is to be found in the left column, and the "3" in the top row. Point out that $1 \square 3 = 5$ and $3 \square 1 = 7$, so it is necessary to be careful about the order in which elements are written.

Some examples for class discussion are given below.

Example 1. Set: The counting numbers.

Rule of Procedure: Given any two elements, take twice the first and add three times the second. This is an operation, but it is not commutative and it is not associative.

Example 2. Set: The counting numbers.

Rule of Procedure: Given any two elements, take twice one of them and add three times the other.

This rule does not define an operation since a "definite thing" is not always determined. For instance, in combining 2 and 3, we are allowed to form either $2 \cdot 2 + 3 \cdot 3 = 13$, or $2 \cdot 3 + 3 \cdot 2 = 12$. The result of an operation applied to two elements must be unique, that is, there can be one and only one answer. It could also be seen from a table that this rule does not describe an operation. The table would have more than one entry in some places (everywhere except on the diagonal from upper left to lower right).

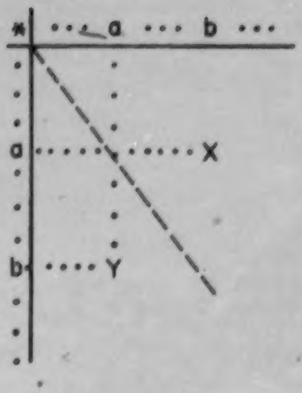
Example 3. Set: The whole numbers.

Rule of Procedure: Given any two elements, divide the first by the second.

This rule does not define an operation, since division by zero is impossible. The elements 2 and 0 cannot be put together in that order. Notice that, in the order 0 and 2, they can be combined (the result is zero). It could also be seen from a table that this rule does not describe an operation. The table would have some of the spaces blank (the column with "0" at the top would be blank).

Discussion of Exercises 12-3

2. In the schematic diagram at the right, a * b is to be entered at position X, and b * a is to be entered at position Y. These two positions are symmetrically located with respect to the diagonal from upper left to lower right. (The elements are arranged in the same order in the top row and left column.) If an operation is commutative, its table will be symmetric about this diagonal, and conversely.



3. Bring out by discussion that, to prove an operation *is* associative, requires testing every triple; one example would be sufficient to prove an operation *is not* associative. This logical point has been illustrated several times previously, and should be re-emphasized here. To prove associativity for an operation by examining all the cases is almost always a long process. Each student should check 2 or 3 cases and if all of them are satisfactory, the following statement can be made: "This operation appears to be associative, but we are not really sure."

4. In making a table for each of these operations, arrange the elements of the set in the same order in the top row and left column. Compute and fill in as many entries in the table as needed to see the pattern. Associativity can be decided from known properties

of the counting numbers.

(a) This operation is not completely described. If the two given numbers are equal, there is no smaller one, of course. Bring out, by class discussion, that, if the two numbers are the same, the result of the operation should be defined as that same number.

(b) and (c) Same as (a) above. Consider the case where the two numbers are the same and decide that the result of the operation will be that number.

5. and 6. The students will need help in beginning these problems. The successive steps are as follows: (1) Choose a set (each pupil may have a different set, but it is better not to have too many elements in each set so the problem will not be too long). Suppose the set {1, 2, 3} is chosen. (2) Make the framework for a table as shown at right. The elements in the set which was chosen in (1) will appear in the left column and top row. Arrange them in the same order. (3) Choose a symbol, such as *, for the operation and put it in the upper left-hand corner of the framework; make up a name to go with it, such as "star." (4) Fill in the table. Emphasize that the names of any objects whatever may be placed in the body of the table - it is not necessary that these objects be elements of the set chosen in (1). If the operation is to be commutative, the table must be symmetric about the diagonal from upper left to lower right. If the operation is not to be commutative, the table must not be symmetric.

*	1	2	3
1			
2			
3			

7. Here, a way to write the information is to arrange the elements and the corresponding results of the operation in two rows or columns. Usually some symbol (such as "x") is used to denote an element of the set and a different symbol (such as "y" or, in this case, "x^3") denotes the corresponding result of the operation. The table is given in two columns in the answers. It could also be written in two rows as shown below.

x	0	1	2	3	4	5	6	7	8	9	10
x^3	0	1	8	27	64	125	216	343	512	729	1000

Notice that a unary operation requires only a one-dimensional table, a binary operation requires a two-dimensional table and a ternary operation would require a three-dimensional table.

Answers to Exercises 12-3

1. (a) 1
   (b) 6
   (c) 8
   (d) 7
   (e) 2
   (f) 3
   (g) 1
   (h) 1
   (i) 8
   (j) Not possible; 1 □ 2 = 4 and 1 □ 4 is not defined since 4 does not appear in the top row.
   (k) 3
   (l) 3

2. (a), (b), (d), (e). The table must be symmetric about the diagonal from upper left to lower right. See discussion, Exercises 12-2.

3. There is no short-cut method; to prove associativity each triple of elements must be combined in the two ways and the corresponding results must be equal. The operations of Tables (a) and (d), (e) are associative; those of Tables (b) and (c) are not. See discussion, Exercises 12-3.

4. See discussion, Exercises 12-3. The operation symbols are omitted in the following tables:

(a)

	26	27	28	...	74
26	26	26	26	...	26
27	26	27	27	...	27
28	26	27	28	...	28
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
74	26	27	28	...	74

Commutative: Yes

Associative: Yes

(b)

	501	502	503	...	535
501	501	502	503	...	535
502	502	502	503	...	535
503	503	503	503	...	535
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
535	535	535	535	...	535

Commutative: Yes

Associative: Yes

(c)

	2	3	5	7	11	...
2	2	3	5	7	11	...
3	3	3	5	7	11	...
5	5	5	5	7	11	...
7	7	7	7	7	11	...
11	11	11	11	11	11	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

Commutative: Yes

Associative: Yes

(d)

	40	42	44	...	60
40	40	40	40	...	40
42	42	42	42	...	42
44	44	44	44	...	44
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
60	60	60	60	...	60

Commutative: No

Associative: Yes

(e)

	1	2	3	...	49
1	3	4	5	...	51
2	5	6	7	...	53
3	7	8	9	...	54
.	.	.	.	...	.
.	.	.	.	...	.
49	99	100	101	...	147

Commutative: No

Associative: No

(Try the triple
1, 2, 3.)

(f)

	1	2	3	4	5	6	...
1	1	1	1	1	1	1	...
2	1	2	1	2	1	2	...
3	1	1	3	1	1	3	...
4	1	2	1	4	1	2	...
5	1	1	1	1	5	1	...
6	1	2	3	2	1	6	...
.	.	.	.	.	.	.	...
.	.	.	.	.	.	.	...
.	.	.	.	.	.	.	...

Commutative: Yes

Associative: Yes

(g)

	1	2	3	4	5	...
1	1	2	3	4	5	...
2	2	2	6	4	10	...
3	3	6	3	12	15	...
4	4	4	12	4	20	...
5	5	10	15	20	5	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

Commutative: Yes

Associative: Yes

(h)

	1	2	3	4	...
1	1	1	1	1	...
2	2	4	8	16	...
3	3	9	27	81	...
4	4	16	64	256	...
.	.	.	.	.	...
.	.	.	.	.	...
.	.	.	.	.	...

Commutative: No

Associative: No

(Try the triple

2, 1, 3;

$(2^1)^3 = 8 \neq 2 = 2^{(1^3)}$)

5. Many answers are possible, of course. The only requirement is that the table be symmetric about the diagonal from upper left to lower right (and that each place in the table be filled uniquely so that the table does describe an operation). See discussion, Exercises 12-2.

*	1	2	3
1	X	Y	Z
2	Y	P	Q
3	Z	Q	R

6. Many answers are possible, of course. The only requirement is that the table must not be symmetric about the diagonal from upper left to lower right (and that each place in the table be filled uniquely so that the table does describe an operation). See discussion, Exercises 12-3.

*	1	2	3
1	X	P	Z
2	P	R	Y
3	Q	P	Q

*7.

x	x^3
0	0
1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512
9	729
10	1000

See discussion, Exercises 12-3.

12-4. Closure

Skills and Understandings

1. To recognize, from the table describing a binary operation, whether or not a set is closed under the operation.

2. To find whether or not a set is closed under a binary operation described in words.

Teaching Suggestions

The concept of closure has been discussed in Chapters 3 and 6 in connection with the usual arithmetic operations. The discussion here should prepare the pupil for consideration of more general systems where the elements may not be numbers.

Bring out, by class discussion, that closure involves two things. (1) It must be possible to put any two (not necessarily different) elements of the set together and (2) the result obtained must always be an element of the set. Material for class discussion is provided by the various parts of Problem 4 of Exercises 12-3.

As with associativity (see discussion of Problem 3, Exercises 12-3), to prove a set is closed under an operation, all cases must be considered; a single counter example would prove that the set is not closed under the operation.

It has been found in some classes that the pupils have difficulty because they expect the concept of closure to be much more difficult than it really is. Perhaps they should be reassured on this point.

The chief purpose of Examples 5 and 6 is to contribute to the understanding of closure by showing what a set must contain if it is to be closed. This is in a way also a preparation for the discussion of the existence of an inverse. Incidentally, the idea of a generator is an important mathematical concept; e.g., all the counting numbers are generated by the single number 1 under addition. This is the principle of mathematical induction: A statement is true for all counting numbers if, first, it is true

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for the number 1 and, second, whenever it is true for a counting number k it is also true for k + 1. In a way we "generate" the truth of the statement for all counting numbers by starting with 1 and proceeding step by step. Some teachers may feel that these two examples are too hard. If they are omitted the following Problems in Exercises 12-4 should also be omitted: 3, 4, 5, 6; also Problem 7 in Exercise 12-6 should be omitted.

Discussion of Exercises 12-4

1. Each table determines a set (the set of elements in the left column and top row), and completely describes the corresponding operation. For a set to be closed under the corresponding operation, each entry in the body of the table must be an element of the set. In Tables (a), (d), and (e) this is true; in (b) and (c), it is not.

*7. From the definition of commutativity in Section 12-3, it must be possible to put any two elements of the set together in either order and the same result must be obtained, but the result of the operation is not required to be an element of the set. In fact, Table (b) of Section 12-3 gives an example of a commutative operation, and the set on which the operation is defined is not closed under the operation.

*8. From the definition of associativity in Section 12-3, it must be possible to put any three elements of the set together in the two ways specified and the same result must be obtained. This means that the set on which the operation is defined must be closed under the operation since, if we can combine a, b, c as (a + b) + c, then certainly a + b must be an element of the set on which the operation is defined. That is, the set is closed under the operation.

9 and 10. The pupils may need help in beginning these problems. The set of elements has been chosen, but each pupil should choose a symbol for his operation and fill in the entries in the table. See discussion of Problems 5 and 6, Exercises 12-3.

Answers to Exercises 12-4

1. The sets of (a) and (d) are closed under the corresponding operations (all the entries in the table appear in the left column and in the top row); those of (b) and (c) are not closed (some entries in tables (b) and (c) do not appear in the left column and in the top row). See discussion, Exercises 12-4.

2.

(a)	Closed	(f)	Not closed	$15 - 35$ cannot be performed
(b)	Closed	(g)	Closed	
(c)	Closed	(h)	Closed	
(d)	Not closed	(i)	Not closed:	$3 + 7$ is not a prime
(e)	Closed	*(j)	Not closed:	$3 + 3 = 11$ (base 5)

3. (a) $\{2, 4, 6, \ldots, 2k, \ldots\}$ where k is a counting number

   (b) $\{2, 2^2, 2^3, \ldots, 2^k, \ldots\}$ where k is a counting number

4. (a) $\{7, 14, 21, \ldots, 7k, \ldots\}$ where k is a counting number.

   (b) $\{7, 7^2, 7^3, \ldots, 7^k, \ldots\}$ where k is a counting number.

5. (a) $1 \odot 1 = 3$, $(1 \odot 1) \odot 1 = 3 \odot 1 = 2$.
$[(1 \odot 1) \odot 1] \odot 1 = 2 \odot 1 = 1$.
If we continue the operation $\odot$, we generate the same set again. Hence the set $\{1, 2, 3\}$ is the sub-set of S generated by 1 under the operation $\odot$.

(b) $2 \odot 2 = 2$, $(2 \odot 2) \odot 2 = 2$.
$[(2 \odot 2) \odot 2] \odot 2 = 2 \odot 2 = 2$.
It is clear that the subset of S generated by 2 under the operation $\odot$ is the subset $\{2\}$.

*6. $\{3, (3 \div 3), (3 \div 3) \div 3, [(3 \div 3) \div 3] \div 3, \ldots\}$
or $\{3, 1, \frac{1}{3}, \frac{1}{9}, \ldots\}$
Yes; 3 and $\frac{1}{3}$ are in the subset of rationals generated by 3 under division. No; $3 \div \frac{1}{3}$ or 9 is not in this subset. Therefore the set is not closed under division and hence it cannot be associative. See the discussion on Problem 8.

7. No; see discussion, Exercises 12-4.

8. Yes; see discussion, Exercises 12-4.

9. Many answers are possible, of course. The only requirement is that each entry in the table belong to the set $\{0, 43, 100\}$ and that each place in the table be filled uniquely so that the table does describe an operation. See discussion, Exercises 12-4.

*	0	43	100
0	0	0	0
43	43	0	43
100	0	43	0

10. Many answers are possible, of course. The only requirement is that at least one entry in the table must not be an element of the set {0, 43, 100} (and that each place in the table be filled uniquely so the table does describe an operation). See discussion, Exercises 12-3.

*	0	43	100
0	0	0	43
43	43	1	0
100	2	0	43

12-5. Identity Element; Inverse of an Element

Skills and Understandings

1. To determine from a table whether there is an identity element for the operation, and if so, what it is.
2. To realize that an element cannot have an inverse unless there is an identity element.
3. To determine from a table which elements have inverses.
4. To find the inverse of an element, if the element has an inverse.

Teaching Suggestions

Let the students experiment with several tables finding identity elements and inverses of elements. Try to lead them to discover that there is an identity element for an operation if, in the table, (1) there is a column exactly like the left column, and (2) there is a row exactly like the top row. The element associated with both will be the same, and will be the identity, because if $ax = x$ and $yb = y$ for all x and y in the set, we may replace x by b and y by a to get $ab = b = a$.

*	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

In the figure the last row and the last column fit the above conditions. 5 is the identity element.

*	3	4	5	1	2
1	4	5	1	2	3
2	5	1	2	3	4
4	2	3	4	5	1
5	3	4	5	1	2
3	1	2	3	4	5

The third column and the fourth row fit the conditions. 5 is the identity.

Lead the students to discover that an element has an inverse if the identity appears in the same relative position in the row as in the column associated with this element when the top row and left column are in the same order.

For example: In the first table the second element in the third row and the third column is the identity element 5. This means that 3 has an inverse. Since 3 was associated with 2 both times to get the identity 5, then 2 and 3 must be inverses. The pairs 1 and 4, and 5 and 5 are seen to be inverses in a similar way.

Notice that the second table has the same elements and the same operation as the first, but that the order of the elements in the left column is different from that in the top row. It is not possible now to use our usual check of symmetry about the diagonal for commutativity. The method of finding the inverse of an element discussed above does not work out either.

The above may be pointed out to the students if you wish. None of the other tables in the chapter will have its top row and left column in different order.

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The teacher should be warned that there is some difficulty about division and subtraction in a non-commutative system. For multiplication b is called the inverse of a if $ab = ba = 1$. This can happen in a non-commutative system. This is such an example, where each element is its own inverse. But the symbol $\frac{2}{3}$ is ambiguous since $3 \cdot x = 2$ has the solution $x = 1$ and $x \cdot 3 = 2$ has the solution $x = 2$. Actually what is usually done for such systems is to multiply by the inverse and not divide at all. For instance we would either have the product $\frac{1}{3} \cdot 2 = 3 \cdot 2 = 3$ or the product $2 \cdot \frac{1}{3} = 2 \cdot 3 = 2$.

·	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

An analagous situation exists for subtraction when addition is not commutative. This can be illustrated in terms of the above example if we replace $\cdot$ by $+$ and $\frac{1}{3}$ by $^-3$.

However, it was felt that such considerations as these were much too complex for inclusion in the text and hence when questions of division or subtraction arise, we restrict the systems to commutative systems.

Answers to Exercises 12-5a

1. (a) In table (a), the identity is 5.
   In table (d), the identity is 2.

   (b) In table (a), the inverse of 1 is 4; of 2 is 3; of 5 is 5.
   In table (b), no element has an inverse.
   In table (c), no element has an inverse.
   In table (d), the inverse of 1 is 3; of 2 is 2.

   Each member of the sets for tables (a) and (d) has an inverse. The operations described by tables (b) and (c) do not have identities so no inverses can exist.

2. (a)

Operation	Identity
(a)	74
(b)	501
(c)	2
(d)	None
(e)	None
(f)	None
(g)	1
(h)	None

(b) The only inverses are those listed below.

(a) 74 is the inverse of 74.
(b) 501 is the inverse of 501.
(c) 2 is the inverse of 2.
(g) 1 is the inverse of 1.

(c) None.

3. No; if there are two identities (P and Q) for a given operation, then consider the result when P is combined with Q. Since Q is an identity, the result must be P. But since P is also an identity, the result must be Q. Thus, P and Q must be the same element since each equals the result of combining P and Q.

Answers to Exercises 12-5b

1. (a) $1x \equiv 1 \pmod{6}$, $x = 1$
$2x \equiv 1 \pmod{6}$, not possible
$3x \equiv 1 \pmod{6}$, not possible
$4x \equiv 1 \pmod{6}$, not possible
$5x \equiv 1 \pmod{6}$, $x = 5$

(b) 1, 5. Each is its own inverse.

2. (mod 5)

b	a	multiplicative inverse of a	b + a	b · (multiplicative inverse of a)
1	2	3	$1 + 2 \equiv 3$	$1 \cdot 3 \equiv 3$
2	2	3	$2 + 2 \equiv 1$	$2 \cdot 3 \equiv 1$
3	2	3	$3 + 2 \equiv 4$	$3 \cdot 3 \equiv 4$
2	3	2	$2 + 3 \equiv 4$	$2 \cdot 2 \equiv 4$
3	3	2	$3 + 3 \equiv 1$	$3 \cdot 2 \equiv 1$
4	3	2	$4 + 3 \equiv 3$	$4 \cdot 2 \equiv 3$
1	4	4	$1 + 4 \equiv 4$	$1 \cdot 4 \equiv 4$
2	4	4	$2 + 4 \equiv 3$	$2 \cdot 4 \equiv 3$
3	4	4	$3 + 4 \equiv 2$	$3 \cdot 4 \equiv 2$
4	4	4	$4 + 4 \equiv 1$	$4 \cdot 4 \equiv 1$

3. (mod 5)

b	a	additive inverse of a	b - a	b + (additive inverse of a)
0	1	4	0 - 1 ≡ 4	0 + 4 ≡ 4
2	1	4	2 - 1 ≡ 1	2 + 4 ≡ 1
4	1	4	4 - 1 ≡ 3	4 + 4 ≡ 3
1	2	3	1 - 2 ≡ 4	1 + 3 ≡ 4
2	2	3	2 - 2 ≡ 0	2 + 3 ≡ 0
3	2	3	3 - 2 ≡ 1	3 + 3 ≡ 1
2	4	1	2 - 4 ≡ 3	2 + 1 ≡ 3
3	4	1	3 - 4 ≡ 4	3 + 1 ≡ 4
4	4	1	4 - 4 ≡ 0	4 + 1 ≡ 0

4. (a) no
(b) no
(c) no
(d) yes, except division by zero.

5. (a) {0, 1, 2, 3, 4, 5}
(b) {1, 5}, {5}
(c) {2, 4}, {1, 5}, {5}

6. (a) {A, B}, {C, D}, {A, D}
(b) yes, D
(c) {C, D}
(d) {C, D}

If you wish, you might bring up the general problem of defining an operation which is inverse to a given operation $*$ defined on a set. If there is an identity element e for $*$, if every element of the set has an inverse element in the set, and if $*$ is associative, then

$$(\text{the inverse of } b) * a$$

could be written $a *' b$. Then $*'$ will be the inverse operation for $*$. Hence

$$a *' b = (\text{the inverse of } b) * a.$$

For example: suppose a and b are rational numbers, $b \neq 0$, and $*$ is the multiplication operation, then $*'$ is division (the inverse operation) and $\frac{1}{b}$ is the inverse of b.

Hence:

$$a \div b = \frac{1}{b} \times a.$$

12-6. What is a Mathematical System?

Here the mathematical system is given an informal definition and is followed by discussion in terms of previous examples and some new ones. Here the teacher should not try to be too formal.

Teaching Suggestions

In Section 12-3, it was pointed out that a table can list a set and describe an operation defined on that set. Thus, a table really describes a mathematical system, and not merely an operation. Illustrate by discussing tables (a) - (e) of Section 12-3, and by showing that each table does describe a mathematical system (a set and one or more operations defined on that set -- in each case, it will be one operation).

In Example 1, Part (c) (egg-timer arithmetic), remind the pupils of the symmetry test for commutativity discovered in Problem 2 of Exercises 12-3. The table for egg-timer arithemetic is symmetric, so the operation is commutative.

Have the class decide on a word for the operation in Table (c) of this section. ("Twiddle" is sometimes used.)

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Answers to Exercises 12-6

1. Each one of Tables (a), (b), (c) describes a mathematical system.
   For Table (a), the set is {A, B}; the operation is o.
   For Table (b), the set is {P, Q, R, S}; the operation is *.
   For Table (c), the set is {Δ, □, ○, \}; the operation is ~ .

2. (a) A
   (b) ○
   (c) ○
   (d) B
   (e) Q
   (f) S
   (g) P
   (h) \
   (i) Δ
   (j) B
   (k) A
   (l) R

3. The operation o is not commutative, since Table (a) is not symmetric.
   The operations * and ~ are both commutative, since both Tables (b) and (c) are symmetric.

4. There is no identity element for the operation o. There is no element e, such that both of the equations A o e = A and B o e = B are correct.
   The element R is the identity element for the operation *. The row of Table (b) with "R" in the left column is the same as the top row, and the column with "R" at the top is the same as the left column.
   The element Δ is the identity element for the operation ~ . The first row and column of Table (c) are the same as the top row and left column respectively.

5. (a) S
   (b) S
   (c) R
   (d) R
   (e) Q
   (f) Q
   (g) \
   (h) \
   (i) \
   (j) \

6. Each of the operations $*$ and $\sim$ seems to be associative since, in each of the cases we have tried, the corresponding expressions are equal. To prove the operations are associative, we would have to examine all cases and show that the corresponding expressions are equal. To prove an operation is not associative, a person would have to find one example where the corresponding expressions are not equal.

7. BRAINBUSTER. (a) The element 2 cannot be combined with 2 by the operation $*$ (that is, $2 * 2$ is not defined).

   (b) $2 * 1$ is not uniquely defined. Many results are possible when 2 and 1 are combined.

   (c) The set given by this table is $\{1, 2, 3, 4\}$. But it is not possible to combine every pair of elements (e.g. 3 and 3). We do not have an operation defined on the set.

12-7. Mathematical Systems Without Numbers

Skills and Understandings

1. To recognize a mathematical system when it is described in words.
2. For systems without numbers: To recognize the elements of the set; to recognize the operation; to recognize an identity element; to recognize the inverse of an element.

Teaching Suggestions

Each pupil should have his own rectangle to manipulate, such as, a 3" × 5" card. Do not use square cards. Be sure that each pupil labels his rectangle correctly so that comparisons between different pupils are possible. Check especially that each corner of the card is labeled with the same letter on both sides. Stress that the card is used only to represent a geometric figure -- a closed rectangular region. Some of the geometric concepts of Chapters 4 and 9 should be reviewed here.

It cannot be repeated too often that the changes of position of a rectangle are the elements of the set in the mathematical system discussed in this section. One of these changes is something that is "done"; that is, it is a physical activity, but it is an element of the set -- it is not the operation of the system. The operation of the system is much more elusive. Any operation defined on the set must be a way of combining any two of these physical activities (changes) to get a definite thing. The particular operation we have chosen combines two of these changes by doing the first one and then the other. The result (definite thing) obtained is one of the changes, but the operation is the way of combining them, that is: First do ..., and then do ... .

For ease in grading pupils' written work it is essential that all students use the same notation in Exercises 5 and *6. One possible notation is described in the answers.

Discussion of Exercises 12-7

3. In proving associativity, "all cases" must be considered. There is one case for each triple of (not necessarily different) elements of the set on which the operation is defined. For the operation ANTH, there are 4 elements in the set, so there will be $4 \cdot 4 \cdot 4 = 64$ triples; that is, 64 cases must be considered to prove the associative property.

5. and *6. For ease in grading written work it is essential that all students use the same notation in these exercises. One possible notation is described in the answers.

Answers to Exercises 12-7

1.

ANTH	I	V	H	R
I	I	V	H	R
V	V	I	R	H
H	H	R	I	V
R	R	H	V	I

2. (a) V
   (b) V
   (c) V
   (d) V
   (e) I
   (f) I
   (g) I
   (h) I
   (i) I

3. (a) Yes

   (b) Yes

   (c) Yes, the operation is associative. A proof would require that 64 cases be checked. Each pupil should check two or three; do not attempt to check all cases. See discussion, Exercises 12-6.

   (d) Yes. I is the identity.

   (e) Yes. Each element is its own inverse.

4. (a)

ANTH	I	F
I	I	F
F	F	I

   (b) Yes

   (c) Yes

   (d) Yes. All cases can be checked (there are 8 cases in all). See discussion of Problem 3, Exercises 12-6.

   (e) Yes. I is the identity element.

   (f) Yes. Each element is its own inverse.

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5.

ANTH	I	R	S	T	U	V
I	I	R	S	T	U	V
R	R	S	I	U	V	T
S	S	I	R	V	T	U
T	T	V	U	I	S	R
U	U	T	V	R	I	S
V	V	U	T	S	R	I

The operation is not commutative (R ANTH T $\neq$ T ANTH R) I is the identity element. Each of I, T, U, V is its own inverse element; R and S are inverses of each other.

*6. Notation:

I: Leave the square in place.

R_1: Rotate clockwise $\frac{1}{4}$ of the way around.

R_2: Rotate clockwise $\frac{1}{2}$ of the way around.

R_3: Rotate clockwise $\frac{3}{4}$ of the way around.

H: Flip the square over, using a horizontal axis.

V: Flip the square over, using a vertical axis.

D_1: Flip the square over, using an axis from upper left to lower right.

D_2: Flip the square over, using an axis from lower left to upper right.

Note: It was suggested that a square card not be used. This problem is included to show why such a suggestion was made.

ANTH	I	R_1	R_2	R_3	H	V	D_1	D_2
I	I	R_1	R_2	R_3	H	V	D_1	D_2
R_1	R_1	R_2	R_3	I	D_2	D_1	H	V
R_2	R_2	R_3	I	R_1	V	H	D_2	D_1
R_3	R_3	I	R_1	R_2	D_1	D_2	V	H
H	H	D_1	V	D_2	I	R_2	R_1	R_3
V	V	D_2	H	D_1	R_2	I	R_3	R_1
D_1	D_1	V	D_2	H	R_3	R_1	I	R_2
D_2	D_2	H	D_1	V	R_1	R_3	R_2	I

I is the identity element. The operation is not commutative (R_1 ANTH H $\neq$ H ANTH R_1).

12-8. The Counting Numbers and the Whole Numbers

This section has problems which lead the pupils to conclude that the counting numbers and the whole numbers each form a mathematical system. It is pointed out that the distributive property with which the pupil is familiar comes from the abstract discussion of this property. The pupils should not be expected to duplicate the abstract definition.

One of the objectives of the section is to show a way to pull together the concept of systems.

Some of the sets of numbers considered in ordinary arithmetic are: the rational numbers, the whole numbers, the counting numbers, the even numbers, etc.

Discussion of Exercises 12-8

4. One possible model of the mathematical system in this exercise is as follows: Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$, $C = \{1, 2, 4\}$, $D = \{1, 2, 3, 4\}$. Then, from the tables in the problem, the operation * is intersection and the operation o is union. Each of these operations distributes over the other.

Answers to Exercises 12-8

1. (a) Since the sum of two counting numbers is always another counting number and the product of two counting numbers is always a counting number, the set is closed under addition and multiplication.

   (b) Both the commutative property and the associative property hold for addition and multiplication.

   Examples: Commutative: $2 + 3 = 3 + 2$;
   $4 \times 6 = 6 \times 4$

   Associative: $3 + (4 + 7) = (3 + 4) + 7$;
   $3 \times (6 \times 8) = (3 \times 6) \times 8$.

   (c) There is no identity element for addition.
   The identity element for multiplication is 1;
   for every counting number n, $n \cdot 1 = n = 1 \cdot n$.

   (d) The counting numbers are not closed under subtraction or division.

2. (a) The set of whole numbers is closed under addition and multiplication.

   (b) Both operations are commutative and associative.

   (c) There is an identity element for addition. It is zero; for any whole number n, $n + 0 = n = 0 + n$. The number 1 is the identity element for multiplication.
   The answers are the same as for 1 (a), (b), (c) except that there is an identity element for addition in the whole number system and not in the counting number system.

3. (a) Three examples are: $2(3 + 4) = (2 \cdot 3) + (2 \cdot 4)$; $5(7 + 10) = (5 \cdot 7) + (5 \cdot 10)$; $1(1 + 1) = (1 \cdot 1) + (1 \cdot 1)$.

(b) Addition does not distribute over multiplication; for example, $2 + (3 \cdot 4) = 14 \neq 30 = (2 + 3) \cdot (2 + 4)$.

4. See discussion, Exercises 12-7.

(a) Yes, here are 3 illustrations that $*$ distributes over $\circ$:

$A * (B \circ C) = A = (A * B) \circ (A * C)$
$B * (B \circ B) = B = (B * B) \circ (B * B)$
$C * (B \circ D) = C = (C * B) \circ (C * D)$

(b) Yes, here are 3 illustrations that $\circ$ distributes over $*$.

$A \circ (B * C) = A = (A \circ B) * (A \circ C)$
$B \circ (B * B) = B = (B \circ B) * (B \circ B)$
$C \circ (B * D) = D = (C \circ B) * (C \circ D)$

5. (a) Closed; commutative; associative; 1 is the identity; only the number 1 has an inverse.

(b) Closed; commutative; associative; no identity; no inverses.

(c) Closed; commutative; associative; 0 is the identity; only the number 0 has an inverse.

(d) Closed; commutative; associative; no identity; no inverses.

(e) Closed; commutative; associative; 0 is the identity; only the number 0 has an inverse.

(f) Not closed; commutative; not associative; no identity; no inverses.

6. (a) Both sets are closed under the operations. Both operations are commutative and associative. Both systems involve the same set.

   (b) The system 5(a) has an identity and 5(b) does not. Also, the sets are different in these two systems.

*7. Many results are possible, of course.

*8. (a) Yes. We are asked to consider the two expressions $a * (b \circ c)$ and $(a * b) \circ (a * c)$, and find whether or not they are always equal. For example, using $a = 8$, $b = 12$, $c = 15$,
$8 * (12 \circ 15) = 8 * 60 = 4.$
$(8 * 12) \circ (8 * 15) = 4 \circ 1 = 4.$

   (b) Yes. We are asked to consider the two expressions $a \circ (b * c)$ and $(a \circ b) * (a \circ c)$, and find whether or not they are always equal. For example, using
$a = 8$, $b = 12$, $c = 15$,
$8 \circ (12 * 15) = 8 \circ 3 = 24$
$(8 \circ 12) * (8 \circ 15) = 24 * 120 = 24.$

12-9. Modular Arithmetic

In this section, the number line is used to provide a picture of how equivalence classes of whole numbers can be developed. At this time it may be wise to re-read the first paragraphs of Section 12-1. We use the term "multiple" to mean "multiple by a whole number."

Problems which may be used for motivation to explain the meaning of modular systems include the ordinary 12-hour clock, the days of the week, and the months of the year. For example, "Today is Tuesday; what day will it be six days from now?" Answer: Monday; this is mod 7. "It is 4:30 o'clock. What time will it be 10 hours from now?" Answer: 2:30; this is mod 12.

Modular arithmetic may be thought of as a mathematical system with two operations. Section 12-1 discussed modular addition and Section 12-2 discussed modular multiplication. The two operations together allow us to use the distributive property; thus, the whole numbers form a system under modular addition and multiplication. In modular arithmetic only a finite number of symbols is needed because infinitely many whole numbers are represented by each symbol.

Other interesting highlights are:

A product of non-zero factors may be zero in some systems.

There may be many replacements for x in a number sentence to make it true.

Answers to Exercises 12-9

1

Mod 5

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Mod 8

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

(Encourage the pupils to look for patterns and to use what they have previously learned about systems to make the tables.)

2. (a) Mod 5: Yes; mod 8: Yes
   (b) Mod 5: Yes; mod 8: Yes
   (c) Mod 5: Yes; mod 8: Yes
   (d) Mod 5: 1; mod 8: 1
   (e) Mod 5: 1 and 4 are their own inverses; 2 and 3 are inverses of each other; 0 has no inverse.
   Mod 8: Only 1, 3, 5, 7 are inverses; each is its own inverse.
   (f) Mod 5: Yes; mod 8: No. $2 \times 4 \equiv 0 \pmod{8}$, $4 \times 2 \equiv 0 \pmod{8}$, $4 \times 4 \equiv 0 \pmod{8}$, $4 \times 6 \equiv 0 \pmod{8}$, $6 \times 4 \equiv 0 \pmod{8}$.

3. (a) 3
   (b) 2
   (c) 6, 8, 12, 24
   (d) 4, 8

4. (a) 2
   (b) 0
   (c) 5
   (d) 0
   (e) 4
   (f) 1
   *(g) 1. Any power of 6 ends in 6.

5. (a) 4
   (b) 2
   (c) 1
   (d) 3

6. (a) 4, 4
   (b) 1, 1
   (c) 3, 3
   (d) yes

7. (a) 0, 0
   (b) 2, 0
   (c) 0, 1
   (d) no

8. (a) 6
   (b) 3, 7
   (c) 0, 4
   (d) 2
   (e) 0
   (f) 0
   (g) 9
   (h) Not defined in this system.

9. (a) 4; What number added to 3 gives 7?

(b) 4

(c) 7

*(d) 7

10.

-	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

The set is closed under subtraction mod 5.

11. (a) 3, 8, 13 and others (add 5)

(b) 3, 7, 11 and others (add 4)

(c) 0 and all multiples of 5 of the form 5K, K is a counting number.

(d) Any even number

(e) 3, or any odd number greater than 3.

(f) 1, 3, 5 and so on (all odd numbers).

12. (d) Any even number

(f) 1, 3, 5, 7, 9, 11, 13 and so on (all odd numbers)

Sample Questions

Part I. True - False

T 1. Operations can be described by tables.

T 2. A symbol can be made to mean anything providing we define it.

F 3. The identity for multiplication in ordinary arithmetic is zero.

F 4. The identity for addition in ordinary arithmetic is one.

T 5. The additive inverse of 2 in the mod 4 system is 2.

T 6. In ordinary arithmetic, with the set composed of all the rational numbers except zero, the inverse of division is multiplication.

F 7. All mathematical systems are sets of numbers.

F 8. In mod 5 arithmetic, $0/3 \equiv 2$ (mod 5).

T 9. The set $\{0, 1, 2, 3\}$ is closed under subtraction mod 4.

Part II. Computation

Find the sums:		Answers:
1.	(9 + 2) (mod 12)	11
2.	(5 + 4 + 3) (mod 6)	0
Find the differences:		
3.	(5 - 2) (mod 6)	3
4.	(3 - 5) (mod 7)	5
Find the products:		
5.	[(3 + 7) × 6] (mod 9)	6
6.	3^2 (mod 8)	1

Find the quotients:	Answers:
7. $\frac{2}{3}$ (mod 5)	4
8. $\frac{0}{7}$ (mod 11)	0

Part III. Multiple Choice

The table below describes a mathematical system. It is to be used in answering questions 1, 2, and 3 below.

o	A	B	C	D
A	C	D	A	B
B	D	A	B	C
C	A	B	C	D
D	B	C	D	A

1. Which one of the following statements is true? (Answers are starred).

   A. The set {A, B, C, D} is not closed with respect to the operation o.

   *B. The operation o is commutative.

   C. The operation o does not have an identity element.

   D. The operation o is not associative.

   E. None of the above.

2. The identity for the operation o is:

   A. D

   B. B

   *C. C

   D. Both A and B

   E. None of the above.

3. In the mathematical system:

A. Only B has an inverse.

B. Only D has an inverse.

C. Only A and C have inverses.

D. None of the elements has an inverse.

*E. All the elements have inverses.

4. For what modulus m is $2 - 5 \equiv 4 \pmod{m}$ true?

A. Mod 9

B. Mod 6

C. Mod 8

*D. Mod 7

E. None of the above.

5. For the system consisting of the set of odd numbers and the operation of multiplication:

A. The system is not closed.

B. The system is not commutative.

C. The system has no identity element

*D. None of the above is correct.

E. All of the above are correct.

6. For the system consisting of the set of even numbers and the operation of addition:

A. The system is not closed.

*B. The system has an identity element.

C. The system has an inverse for addition for each element.

D. All of the above are correct.

E. None of the above is correct.

7. A mathematical system consists of several things. Which of the following is always necessary in a mathematical system?

A. Numbers

B. An identity element

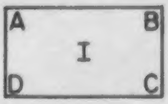
C. The commutative property

*D. One or more operations

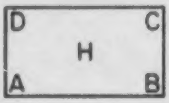
E. None of the above

Use the mathematical system as described below in answering Questions 8, 9, and 10. The set of elements in our system is the set of changes of a rectangle.

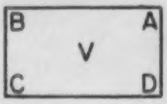
The elements are:



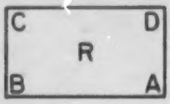
I means leave alone.



H means flip on the horizontal axis



V means flip on the vertical axis



R means turn halfway around its center

The following is an illustration of our operation * ;

V * H means do change V and then do change H.

Thus V * H = R.

8. H * H is:

A. H

*B. I

C. R

D. V

E. None of the above.

9. I * R is:

*A. R

B. V

C. I

D. H

E. R * H

10. (H * V) * V is:

A. I

B. V * V

*C. H * I

D. V

E. None of the above.

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Chapter 13

STATISTICS AND GRAPHS

Introduction

Traditional seventh grade material gives some time to graphs but treats them as an isolated topic. In this text, graphs are introduced as an integral part of statistics. Statistics, as a topic for seventh graders, is new. Only the most elementary phases of statistics are given but the student is expected to become aware of this branch of mathematics. In addition to the three common types of statistical graphs presented in the text, the concept of mean, median and mode as measures of central tendency, and of average deviation and range as measures of spread are introduced. The emphasis is on understanding the meanings of these measures rather than developing skill in finding the measures.

Students of this age are beginning to find a need for understanding elementary statistics. Work in social studies includes current events and some of this material contains graphs and statistics that students should understand. School drives often make use of student-made graphs.

The collection and organization of data is touched upon and opportunity for organization is included in the exercises. The teacher may find it of value to have students collect, organize and graph data of their own. Every school provides its own material. The number of students enrolled, the size and number of class sections, and the number of students that participate in various activities are some areas in which statistical data are usually available.

The interpretation and ability to read graphs are probably of more importance than the ability to make graphs. The teacher should stress this phase of the work with graphs. Making graphs is one activity that helps the student learn to read graphs.

The text provides some work in finding the mean, median, mode, range and average deviation as an aid to understanding the meaning of these terms. The work on sampling is very general. Pupils may be interested in reporting examples of sampling found in newspapers and news magazines.

Additional material that can be used is found in the annual reports of large corporations and in the material released by the National Industrial Conference Board. The Conference Board will send material at your request. (See the Bibliography.) You and some of your students might be interested in the informative and pleasant book, How to Lie with Statistics.

This chapter should require about 10 days to teach.

13-1. Gathering Data

This section introduces the word "data" and develops its meaning. The organization of data and reading a statistical table are utilized both as ideas and as a means in understanding data.

Answers to Exercises 13-1

1. The general trends in the data show that the population has been increasing since 1790. The per cent of increase has been less since 1860 than before that time.
2. From 1800 to 1810. High immigration rate and beginning of Westward Movement.
3. From 1930 to 1940. The depression of the 1930's.
4. Population increase was due in part to large number of immigrants from Ireland. Famine was due to failure of potato crop in Ireland in 1845, 1846, and 1847.
5. 26.0

6. These may be arranged from largest to smallest or smallest to largest.

100%
98%
95%
91%
88%
86%
84%
83%
80%
80%
78%
77%
77%
75%
75%
72%
72%
70%
67%
61%

7. This information can be found in almanacs or local publications.

8. (a)

1935	34,956	1942	28,781	1949	188,317
1936	36,329	1943	23,725	1950	249,187
1937	50,244	1944	28,551	1951	205,717
1938	67,895	1945	38,119	1952	265,520
1939	82,998	1946	108,721	1953	170,434
1940	70,756	1947	147,292	1954	208,177
1941	51,776	1948	170,570	1955	237,790

(b)

23,725
28,551
28,781
34,956
36,329
38,119
50,244
51,776
67,895
70,756
82,998
108,721
147,292
170,434
170,570
188,317
205,717
208,177
237,790
249,187
265,520

9. Number of automobiles sold by the manufacturer:

1910	181,000	1930	2,787,456	1950	6,665,863
1915	895,930	1935	3,273,874	1955	7,920,186
1920	1,905,560	1940	3,717,385		
1925	3,735,171	1945	69,532		

13-2. Broken Line Graphs

Broken line graphs are utilized whenever the change that occurs in some item is to be emphasized. The teacher should be sure students know how to read these graphs. One of the best ways of learning to read graphs is to make them.

Some observations on making graphs are noted below. It would be helpful if the student listed these points before making any graphs. The points are:

1. Plan before making any marks on the paper. This includes planning room for title, scales and any names needed.
2. Make graphs as large as space permits.
3. Print.
4. Use rulers to draw lines that should be straight and to enclose the graph with line segments for a finished appearance.
5. Use a suitable scale. This is found by dividing the largest number to be graphed by the number of units available on the scale.
6. Line graphs should be started at the left edge.

Values between points should be interpreted cautiously. Changes are not instantaneous but since they are irregular, no certainty can be attached to a reading between points. These readings will be fair approximations much of the time.

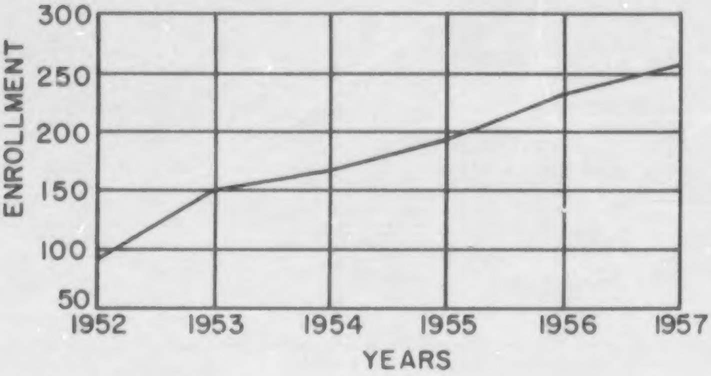
Answers to Class Discussion Questions 13-2

1. More between 1900 and 1910 than between 1800 and 1810. This is seen more easily from the graph than from the table of data.
2. No.
3. The piece of the broken line from 1810 to 1820 would be horizontal.
4. 1945: 140 million; 1895: 68 to 70 million. Increase 70 to 72 million.
5. About 170 million.

Answers Exercises 13-2

1. (a) 20
   (b) 25
   (c) 10,000
   (d) 30 or 50
   (e) 100,000

2. Enrollment in Franklin Junior High School, 1952-1956



3. (a) Graph is on the next page.

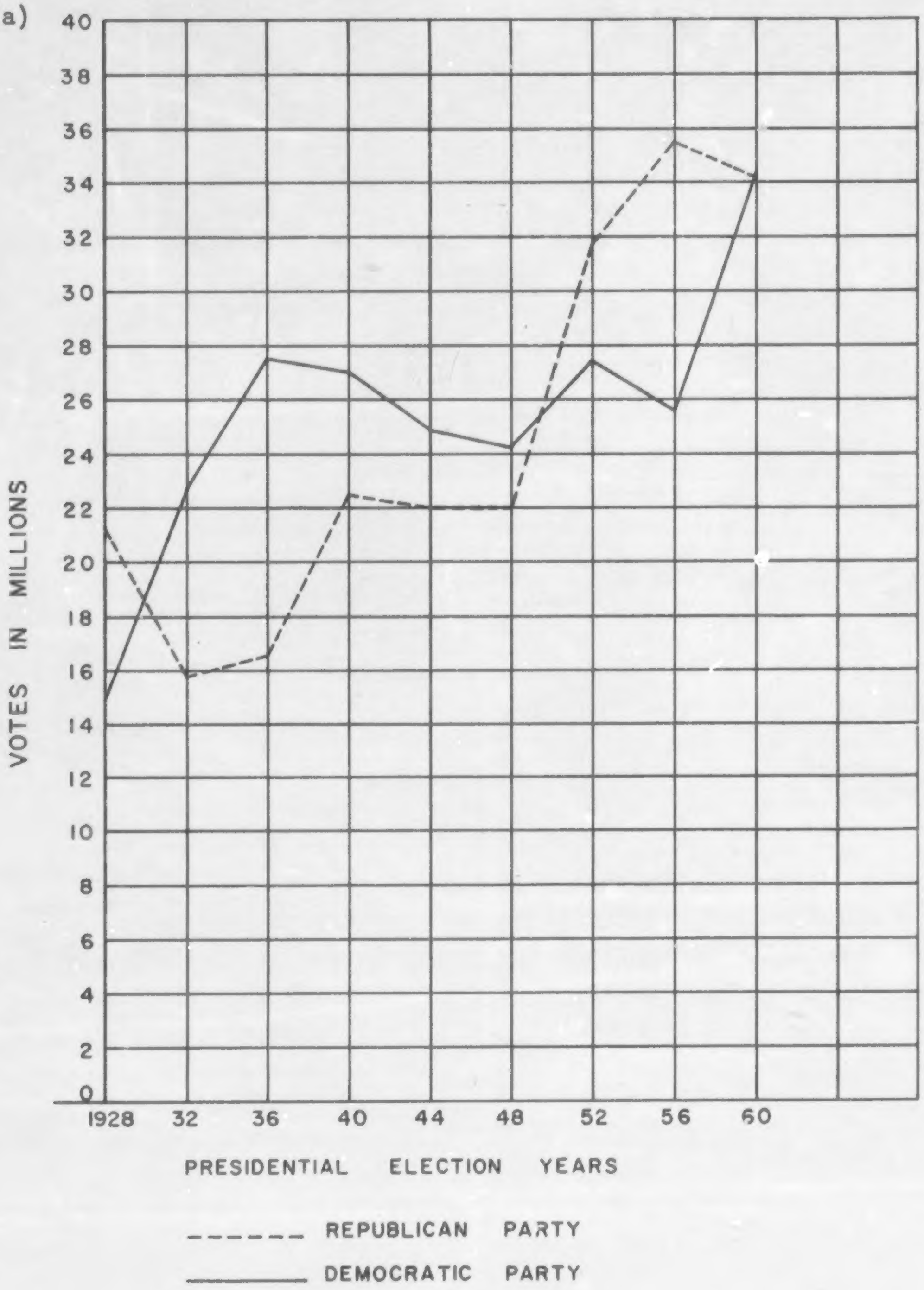
   (b)

(D) Democrat
(R) Republican

Year	Elected	Unsuccessful Candidate
1928	Hoover (R)	Smith (D)
1932	F. D. Roosevelt (D)	Hoover (R)
1936	F. D. Roosevelt (D)	Landon (R)
1940	F. D. Roosevelt (D)	Willkie (R)
1944	F. D. Roosevelt (D)	Dewey (R)
1948	Truman (D)	Dewey (R)
1952	Eisenhower (R)	Stevenson (D)
1956	Eisenhower (R)	Stevenson (D)
1960	Kennedy (D)	Nixon (R)

   (c) $\frac{22.3 + 26.8}{131.7} = 37.3\%$

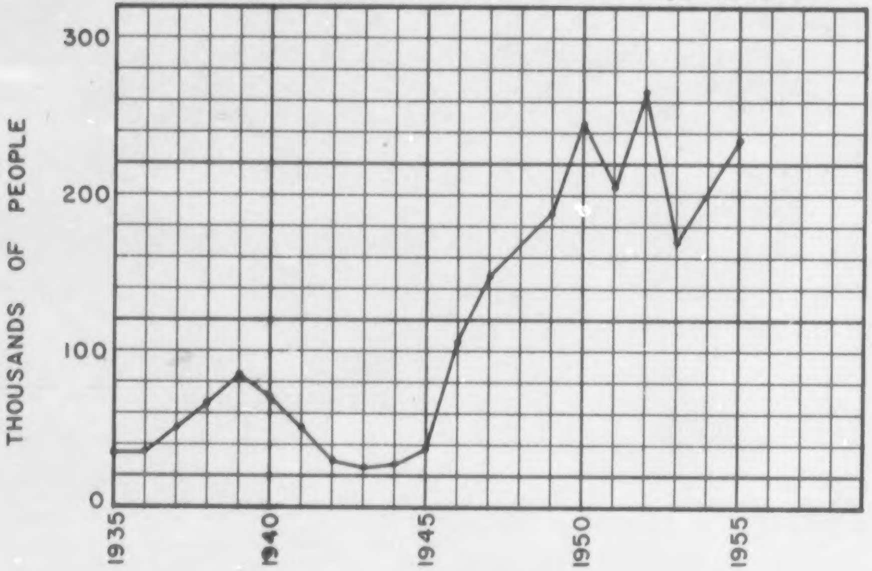
3(a)


VOTES IN MILLIONS
40
38
36
34
32
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
1928
32
36
40
44
48
52
56
60
PRESIDENTIAL ELECTION YEARS
------ REPUBLICAN PARTY
——— DEMOCRATIC PARTY

4. This graph is dependent on local statistics.

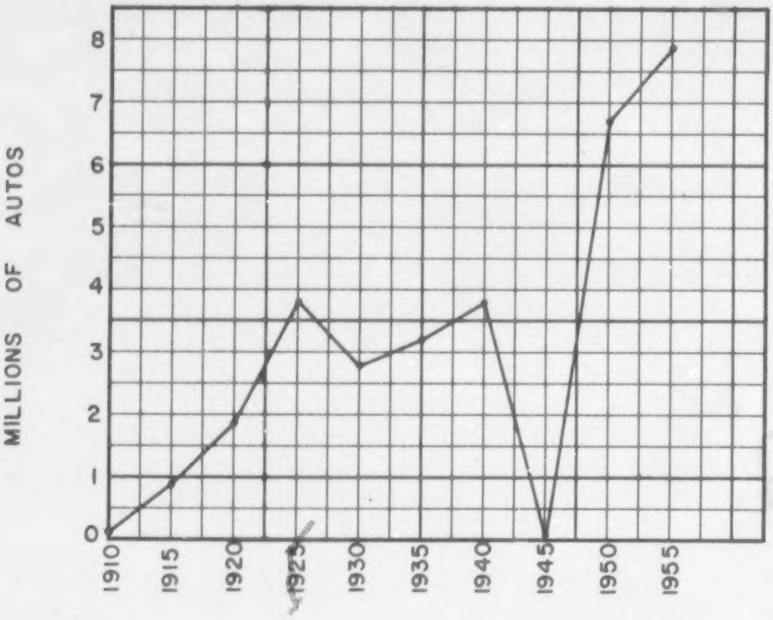
5.

Immigrants Admitted to U.S. Annually, 1935-55



6.

Factory Sales of Autos 1910-55



13-3 Bar Graphs

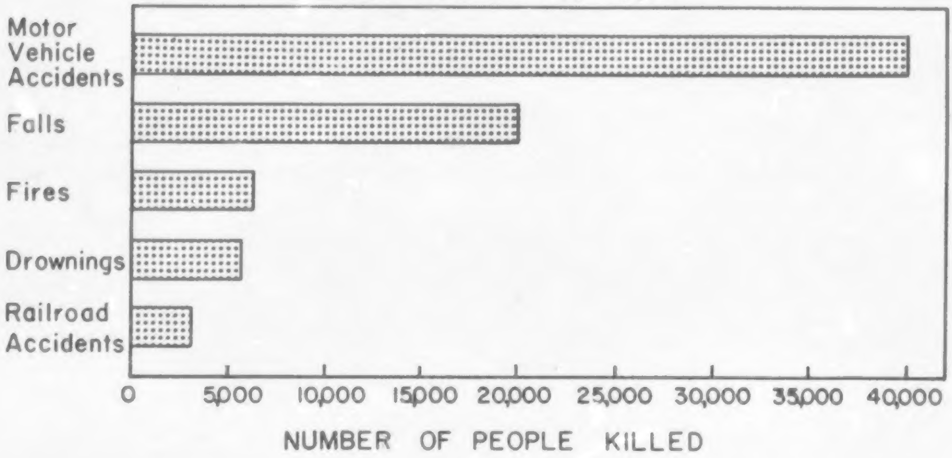
Bar graphs are used to compare data about similar items. Since data that indicate change can be considered as a comparison of similar items, most line graphs could also be shown as bar graphs. Bar graphs cannot always be displayed as line graphs. It is suitable to show the growth of one school in either type of graph but the graph of the enrollment of different schools in a district is suitable only for a bar graph. Line graphs emphasize change; bar graphs emphasize comparison.

Principles of graph construction that were given for line graphs are applicable to bar graphs. One additional point needs to be made. Bars should be the same width; spaces should be the same width but it is not necessary for the spaces to be the same width as the bars. Color adds a great deal to graphs.

Answers to Exercises 13-3

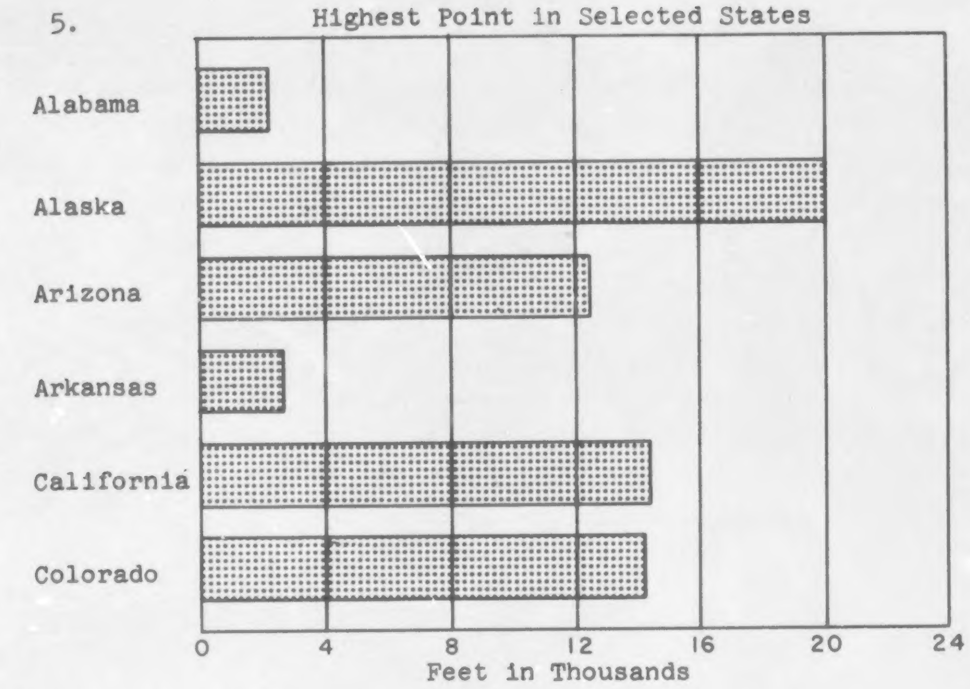
1. 3,260,000 × \$3.25 = \$10,595,000
2. 1959
3. Between 1958 and 1959. Table shows increase is 368,000.
4.

Accidental Deaths, 1956

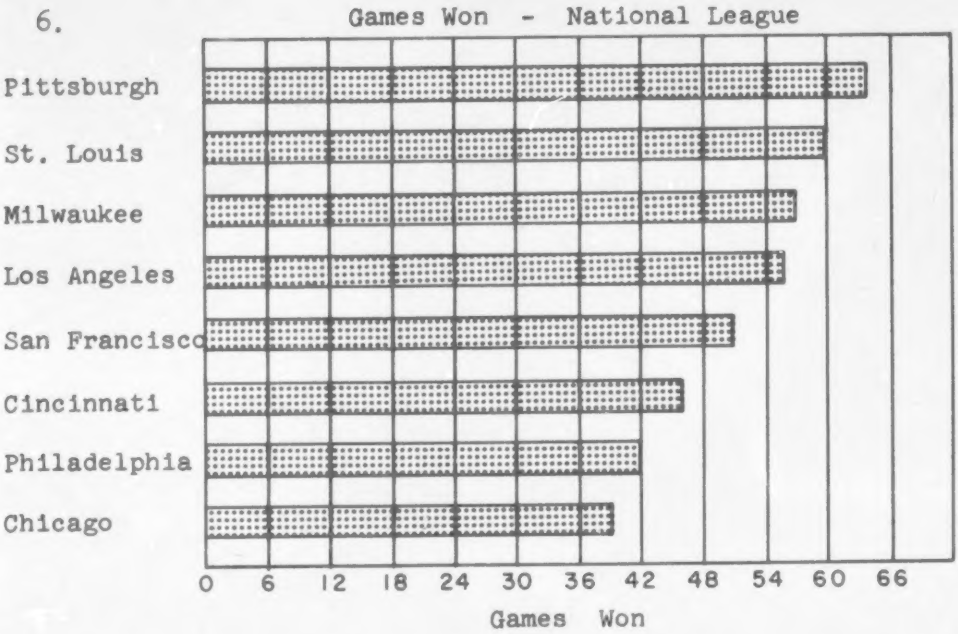


[page 588-590]

5.


Highest Point in Selected States
Alabama
Alaska
Arizona
Arkansas
California
Colorado
0
4
8
12
16
20
24
Feet in Thousands

6.


Games Won - National League
Pittsburgh
St. Louis
Milwaukee
Los Angeles
San Francisco
Cincinnati
Philadelphia
Chicago
0
6
12
18
24
30
36
42
48
54
60
66
Games Won

[page 590]

13-4 Circle Graphs

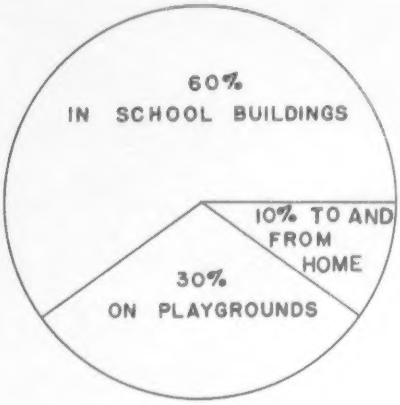
Circle graphs can be used only when the data are concerned with the whole of something and the divisions of this item. It is possible to consider such data as a comparison of parts and use a bar graph to show the relationships. The comparison of parts is shown equally well by circle graphs. Circle graphs emphasize the unity of the item graphed.

The students may need some review both in the use of a protractor and in using percents before they will be able to work with circle graphs.

It may be advisable to have students select from the four graphs in the Exercises. Possibly, some of the class could do two while the rest of the class do the other two.

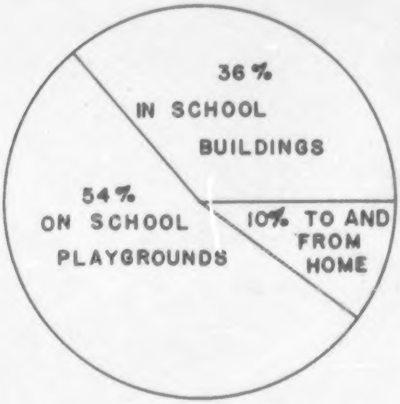
Exercises 13-4

1. School-Related Accidents, 1949



2.

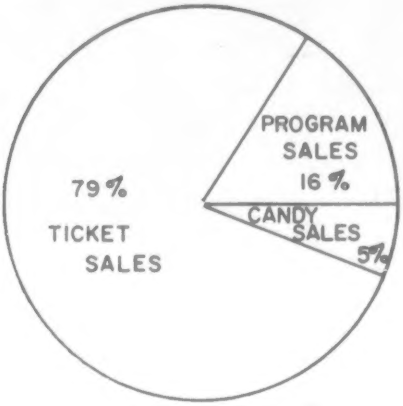
School-Related Accidents, 1956



3.

Money for Stage Curtains

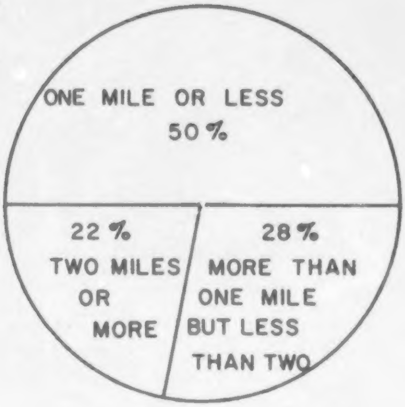
Percent	Number of Degrees
79	284
16	58
5	18



4.

Homes of Washington Jr. High Pupils in Relation to School

Percent	Number of Degrees
50	180
22	79
28	101



13-5 Summarizing Data

The approach to descriptive statistics used here is one of trying to describe a set of numbers by just two numbers, one to give an idea of the magnitudes of the numbers and the other to show how the items vary or scatter.

The averages (mean, mode, and median) give the idea of the magnitude of the numbers in the set; the range and the average deviation from the mean show how the numbers vary or scatter.

The arithmetic mean and the median are used so frequently in newspapers, magazines, and other media that it is most important for everyone to understand thoroughly the two concepts.

The short-cut of Problem 6 in Exercises 13-5c is a standard short-cut which the pupils may find useful. No attempt at a rigorous proof for it should be made.

In statistics a deviation is technically defined so as to involve positive and negative numbers but there is no need to introduce negative numbers here since in computing the average deviation one averages the absolute values of the deviations.

Answers to Exercises 13-5a

1. Mode: 85
2. (a) Arithmetic mean: $\frac{913}{11} = 83$
   (b) Median: 85
3. (a) Mean: \$6100
   (b) 3
   (c) 7
   (d) No, because the mean is larger than such a large percentage of the salaries.
   (e) If there is an even number 2n of items (as there are 10 in this problem) the median after arranging the items according to size is taken to be the average between the nth and (n + 1)th items.

$$\frac{\$5000 + \$5500}{2} = \$5250$$

   (f) Median is better than mean, since the mean gives the impression that the salaries are higher than they are. The mean is affected by the large salary of \$12,500, but the median is not.
4. (a) Mean: 56.5
   (b) Median: 49

Answers to Exercises 13-5b

1. In making tables of ages of pupils in each grade.

   In tabulating data on distances from school, etc.

2. Median: between 15 and 17.

3.

Temperature	Frequency (No. in temperature range)
50 - 54	1
55 - 59	1
60 - 64	2
65 - 69	2
70 - 74	7
75 - 79	3
80 - 84	3
85 - 89	1
90 - 94	1
Total	21

$\frac{1}{2} \cdot 21 = 10\frac{1}{2}$. Consequently the middle number will be the 11th one. There must be ten temperatures below the median temperature and ten above. In order to find the temperature which corresponds to 11, we count from the top of the frequency column. To obtain 11 we must use 5 of the frequency 7. Hence the median is nearer 74 than 70. This may be checked by counting from the bottom of the table.

Answers to Exercises 13-5c

1. In 1953 and 1954.

2. 1951

3. Mean is 44.5. The deviations from it are 0.5, 0.5, 1.5, 1.5. The average deviation is $\frac{4}{4}$ or 1.

4. Average or mean is 83.4

   Deviations from the mean are 1.6, 1.4, 4.6, 7.4, 6.6, 0.6, 3.4, 1.4, 0.6, 0.4 .
   Average of these deviations from the mean is 2.8.

5. Mean is 84.
   Deviations are 10, 0, 16, 10, 14, 14, 12, 0, 8, 12.
   Average deviation from the mean is 9.6.

*6. Suppose the guessed average is 45. Then the deviations of the scores are 5, 2, 1 and 2, 3, 4, 6. The sum of the deviations for scores less than 45 is 8. The sum of the deviations for scores greater than 45 is 15.
$\frac{15-8}{7} = \frac{7}{7} = 1$. Since the sum of the deviations for the scores greater than 45 is more than the sum of the deviations for the scores smaller than 45 we add 1 to the guessed mean to get the correct mean, 45 + 1 = 46.

13-6 Sampling

Although the theory of random sampling is much too complicated to be presented to the pupils, sampling has become such an important part of engineering, business, politics, and other fields that it is important for everyone to have some acquaintance with the idea of sampling.

Members of the class may find it interesting to bring in the results of some of the public opinion polls reported in newspapers and magazines, especially around election time. You may wish to discuss some examples of biased sampling.

The numbers found in Problem 2 are close to the actual results. It is impossible to come closer with the percents given to only the nearest tenth of a percent, since 0.1 % makes a difference of more than 48,000 votes. For your information, Truman received 24,105,812; Dewey, 21,970,065; Thurmond, 1,169,021; Wallace, 1,157,172 votes.

Answers to Exercises 13-6

1. The nature of this problem results in answers that vary.

2.	(a)	Truman	21,730,000
		Dewey	24,170,000
		Thurmond	980,000
		Wallace	1,950,000
	(b)	Truman	24,120,000
		Dewey	21,980,000
		Thurmond	1,170,000
		Wallace	1,170,000

[pages 606-607]

Sample Test Items

Use these data for Problems 1-5

Gross Debt of the United States for the Years 1861 to 1955
(Five-year Periods)

(a) Period	(b) Millions of Dollars	(c) Billions of Dollars	(d) Deviations from Mean
1861 - 65	2,678	3	
1866 - 70	2,436	2	
1871 - 75	2,156	2	
1876 - 80	2,091	2	
1881 - 85	1,579	2	
1886 - 90	1,122	1	
1891 - 95	1,097	1	
1896 -1900	1,263	1	
1901 - 05	1,132	1	
1906 - 10	1,147	1	
1911 - 15	1,191	1	
1916 - 20	24,299	24	
1921 - 25	20,516	21	
1926 - 30	16,185	16	
1931 - 35	28,701	29	
1936 - 40	42,968	43	
1941 - 45	258,682	259	
1946 - 50	257,357	257	
1951 - 55	274,374	274	

1. (a) What is the mode of the set of figures in column (c)? In other words, which gross debt appears most often?

   (b) Does this number tell you very much about the total debt picture of the United States Treasury since 1861?

2. (a) Find the median of the set of figures in column (c) which will give you the median national debt from 1861 to 1955. This means that you will need to find the middle figure between 1 and 274. In order to do this, list the numbers from column (c) in order, and find the middle one on the list by counting up from the bottom or down from the top.

   (b) Does the median tell you much about the total debt over this period of years?
   Do you see that it is necessary to find the mean in order to take into consideration the very large numbers in the table?

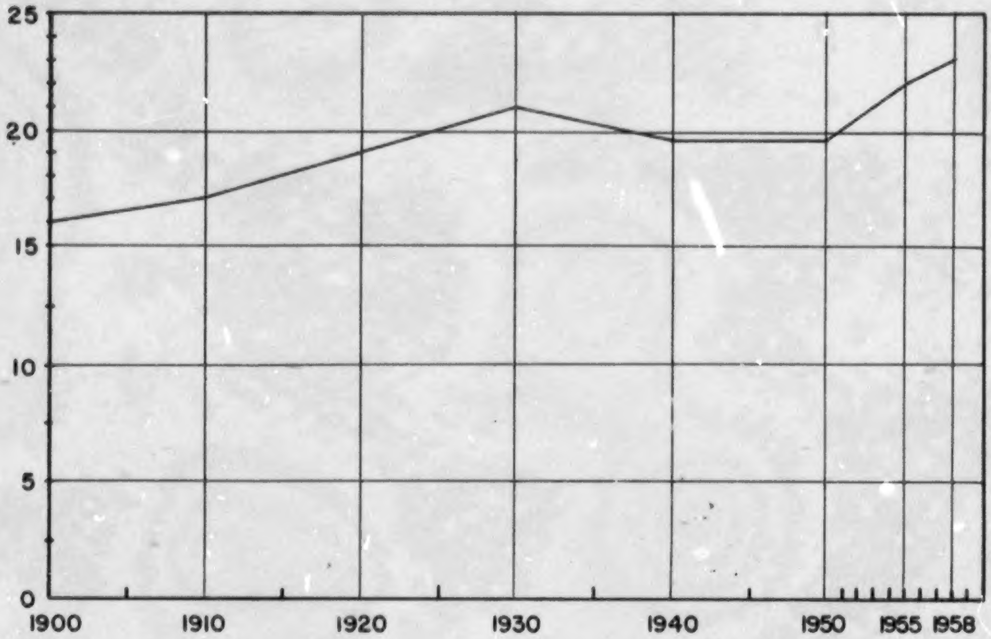
3. Find the mean of the set of numbers in column (c) to the nearest whole number.

4. Find the deviations from the mean and then the average deviation to the nearest whole number.

5. Explain the large deviations above the mean.

6.

Public School Enrollments in United States
Kindergarten and Grades 1--8
(Enrollment is given in millions for the years
shown on the horizontal base line.)



(a) What kind of graph is the graph in this problem?

(b) What was the enrollment in 1910? 1920? 1930? 1958?

(c) In which two years was the enrollment the same?

(d) Estimate the enrollment in 1925.

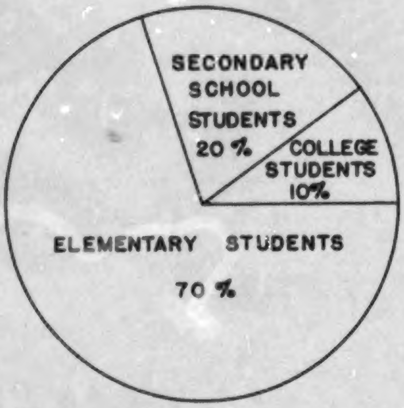
(e) In which ten-year period did the enrollment decrease?

(f) If the graph continues in a straight line for the years 1955, 1958, and 1960, what would be the predicted enrollment in 1960?

7.

School Enrollment

Year: 1950



(a) What fractional part of the total enrollment is in elementary school? in secondary school? in college?

(b) Account for such a large group of children enrolled in elementary school.

(c) If you were asked to draw this circle graph, how would you find the size of each closed curve?

(d) If there were about 29,000,000 students enrolled in school in 1950, how many were in the elementary grades? in college? in secondary school?

Answers to Suggested Test Items

1. (a) Mode = 1 (b) No.

2. (a) Median = 1 (b) No.

3. Mean = 49, to nearest whole number of billions.

4. Deviations from mean are 46, 47, 47, 47, 47, 48, 48, 48, 48, 48, 48, 25, 28, 33, 20, 6, 210, 208, and 225, respectively. Average deviation is 67.2.

5. World War II; Defense Program since World War II.

6. (a) Broken line graph.
   (b) 17 million, 19 million, 21 million, 23 million.
   (c) 1940, 1950.
   (d) About 20 million.
   (e) 1930 - 1940.
   (f) 24 million.

7. (a) $\frac{7}{10}$, $\frac{1}{5}$, $\frac{1}{10}$.
   (b) Compulsory school attendance and increase in birth rate.
   (c) 36°, 72°, 252°.
   (d) 20,300,000; 2,900,000; 5,800,000.

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This organization will send business statistics and graphs free.

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Chapter 14

MATHEMATICS AT WORK IN SCIENCE

14-1. The Scientific Seesaw

Much of the science that is taught in the schools today is descriptive science, and it omits or neglects the studies of a quantitative nature by which one learns to carry out the main steps of a scientific experiment, namely:

(a) The scientist observes what happens and collects and studies the quantitative data.

(b) To explain the facts he has observed, he states his hunch as a hypothesis which expresses the pattern he sees in the data.

(c) He makes predictions, telling what will be seen if certain other observations in the experiment are made.

(d) He goes back to the equipment and tests his prediction. If it works in a reasonable number of cases, he can state his discovery as a theory or principle. No amount of tests will "prove" the principle, but a rather large number of tests will show that the principle is very likely to be true. The principle is usually stated with mathematical terms and symbols.

The purposes of this chapter are:

(a) to illustrate by a simple experiment the typical inductive method of science;

(b) to give pupils experience in collecting mathematical data in an experiment in science;

(c) to help pupils see the importance of the quantitative aspect of the physical experiment;

(d) to furnish applications of some mathematical skills.

The chapter is rather short and probably can be completed in 4 or 5 days.

14-2. A Laboratory Experiment

The equipment is simple. If standard units of known weight are not available, new marbles from the same manufacturer or new pennies can be used. Marbles are colorful and about right as units of weight, but they roll when they are dropped. A meter stick or yardstick is needed to serve as the balance rod or seesaw.

The law of the lever is one that can be discovered rather easily if measurements are carefully made; and yet it is fundamental in science. It is used in much equipment that we see around us such as crowbars, tongs, and scissors.

It is recommended that each pupil take part in the experiment. The teacher may find it profitable to divide the class into groups of four or five pupils so that each group can carry out its own experiment. A special committee of pupils should be appointed in advance to secure the necessary equipment and to prepare it for use.

Square pieces about 8 inches on the side may be cut from the thin plastic covers or bags used in packaging. The four corners may be tied together with sturdy thread which is then looped around the meter stick or yardstick. The stick may be suspended by the same kind of thread. To simplify reading distance measurements, the stick should be balanced at the center point before the bags are suspended. A thumbtack at the proper place will help balance the stick if it is out of balance in the beginning.

The problem of finding a support from which to suspend the meter stick (balance bar) may take some ingenuity to solve. It may be possible to borrow standards from the science department or to manufacture hooks by using coat hangers. As a last resort some teachers have used another stick resting across the backs of two chairs as a support for the balance.

Be sure to give complete and clear instructions in advance so that each pupil understands what is to be done.

In the group of pupils working together on the experiment, two pupils may adjust the distances and the others check the measurements and keep the record. As the experiment proceeds, the pupils may rotate responsibilities so that each has an opportunity to do the complete experiment.

TABLE I

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
w	10	10	10	10	10	10	10	10	10	10
d	12	12	12	12	12	12	12	12	12	12
W	10	20	5	8	15	24	12			
D	12	6	24	15	8	5	10			

As the pupils are doing the experiment and recording measurements in Table I, help them discover two relationships: if the value of w is increased, the corresponding distance, d, from the fulcrum is decreased; also, if a value of the weight is doubled, the corresponding distance from the fulcrum is only half as much; if the weight is made three times as much, the distance is only one-third as much, and so on. The product of w and d as well as W and D in each column in Table I should be approximately 120.

The teacher should be aware that the weight of an object means the gravitational force exerted on it by the earth. Although we have not belabored the distinction between mass and weight, it seems worthwhile to speak precisely in this respect. In particular, the students should be encouraged not to confuse the weight of an object with the object itself, as is frequently the case in ordinary conversation.

Tables II, III, and IV suggest ways of beginning the measurements in order to get a large number of readings and also to help bring out certain relationships. Also, reading the meter stick should increase the pupils' familiarity with this kind of measurement.

TABLE II

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
w	16	16	16	16	16	16	16
d	6	6	6	6	6	6	6
W	16	24	6	12	4	8	48
D	6	4	16	8	24	12	2

TABLE III

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
w	20	40	10	15	30	$7\frac{1}{2}$	50
d	15	$7\frac{1}{2}$	30	20	10	40	6
W	20	20	20	20	20	20	20
D	15	15	15	15	15	15	15

TABLE IV

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
w	18	18	18	18	18	18	18
d	5	5	5	5	5	5	5
W	15	10	30	45	9	6	5
D	6	9	3	2	10	15	18

14-3. Caution: Inductive Reasoning at Work!

After recording all the measurements in their tables, pupils should be able to see and formulate in their own words the law of the lever, namely that the product of the measures of one weight and the corresponding distance from the fulcrum is equal to the product of the measures of the other weight and the corresponding distance from the fulcrum. The formula $wd = WD$ should be discovered by the pupils.

Answers to Exercises 14-3

1. (a) Equal masses are at equal distances from the fulcrum.

   (b) D is half as much as it was.

   (c) D is twice as much as it was. The new D is twice as large as the original D.

   (d) wd = WD. The product of the measures of the distance and of the weight on one side equals the product of the measures of the distance and of the weight on the other side.

2. Table V

w	25	35	12	45	21	15	23	11	$14\frac{6}{7}$	14	10	100	100	100
d	3	4	5	3	5	6	4	8	7	5.5	50	50	500	5,000
W	15	20	7	15	14	20	12	$8\frac{4}{5}$	13	10	100	1000	10,000	100,000
D	5	7	$8\frac{4}{7}$	9	$7\frac{1}{2}$	4.5	$7\frac{2}{3}$	10	8	7.7	5	5	5	5

3. Suggest that pupils check the cases with the smallest numbers.

The purpose of Table V is to provide opportunity for pupils to use arithmetic and apply the law to specific cases. The last four figures in the table show that if a sufficiently long lever is used, an object of tremendous weight can be lifted.

14-4. Graphical Interpretation

Graphing is a skill that needs to be used both in mathematics and science. This experiment gives a set of figures which can be graphed easily.

Be sure that the pupils know how to prepare the axes and how to mark the scale on each axis. They may need some preparatory practice in plotting points and probably will find it difficult to draw a smooth curve through the points that they plot.

The points for which one of the pair of numbers is to be found are: (4, 30); (20, 6); (16, $7\frac{1}{2}$); ($6\frac{2}{3}$, 18).

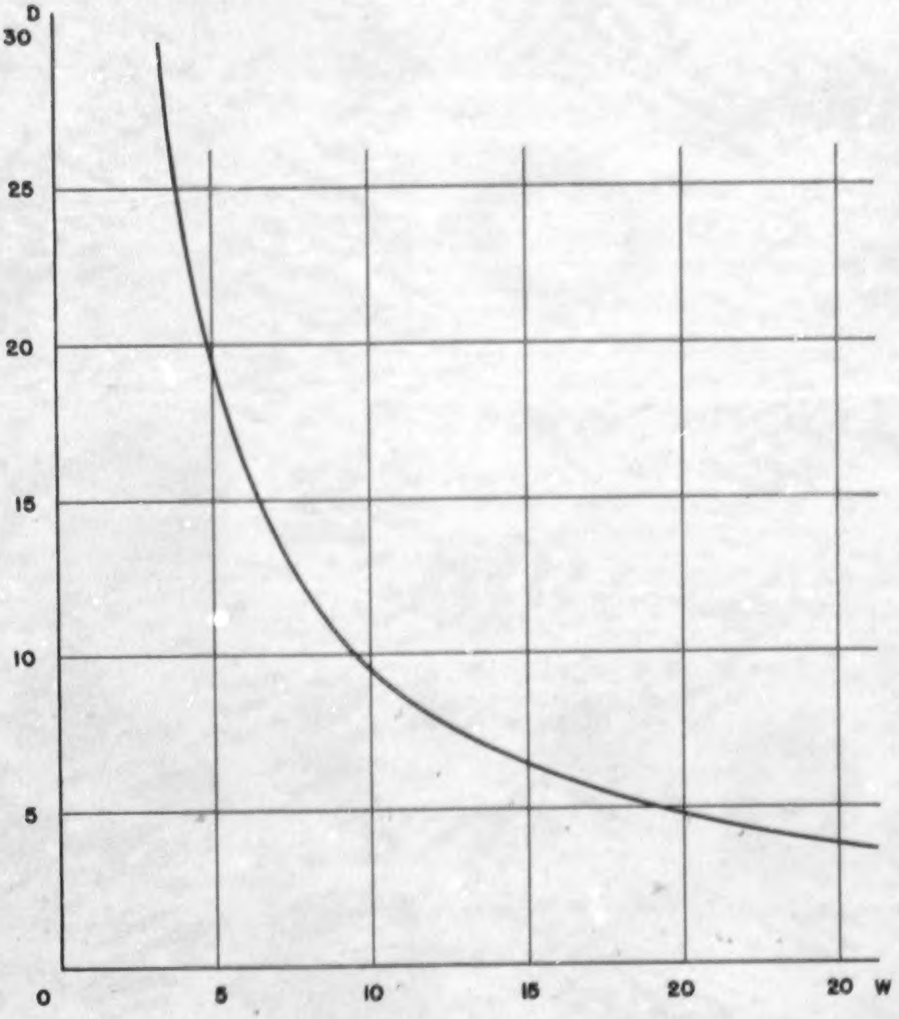
The teacher may wish to follow this graphing with several other graphs, for practice, which yield similar curves, as $WD = 48$ or $WD = 90$. The hyperbola is such an interesting curve that the better students should be encouraged to investigate it and its uses further. It may be appropriate to tell the pupils that the hyperbola is the curve of inverse variation. It results whenever the product of two quantities is a constant as in $WD = 120$, which can also be written $W = \frac{120}{D}$ or $D = \frac{120}{W}$.

Answers to Exercises 14-4

1. (a) As the value of w is increased, the corresponding value of d decreases.

   (b) As d is increased, the corresponding value of the w is decreased. (It may be appropriate to point out that this is an inverse relationship.)

2. (a) (6, 20) (b) (8, 15) (c) (20, 6)
3. (a) (6, 20) (b) (4, 30) (c) (15, 8)
4. (a) (10, 25) is not. (c) (5, 5) is not.
   (b) (15, 8) is. (d) (20, 15) is not.
5. These values are approximations. For example, 17.1, a little more than 17 or approximately 17 are all correct in (a).
   (a) (7, 17.1) (d) (17, 7)
   (b) (8, 15) (e) (5.7, 21)
   (c) (13.3, 9) (f) (23, 5.2)
6. See graph.
7. When W is 20, D is $4\frac{4}{5}$. When D is 12, W is 8.



GRAPH OF WD=96
FROM THE DATA OF TABLE II

8. As the value of w decreases, the corresponding value of d increases.
   As the value of w increases, the corresponding value of d decreases.

9. The curves have similar shapes. Both show inverse variation.

In all this work, help pupils to become aware of the importance of mathematics in science and of the need for mathematical data to carry out experiments.

14-5. Other Kinds of Levers

Other types of levers are simply mentioned in passing. Pupils who are interested may study them individually but it is not in keeping with the spirit of the chapter to spend class time in a thorough investigation of all types of levers.

14-6. The Role of Mathematics in Scientific Experiment

This discussion of the lever does not introduce new mathematics. The intent here is to suggest in a small way some of the many important roles which mathematics plays in our civilization.

In quantitative studies of all kinds, that is, studies in which quantities are important and are carefully measured, the mathematics of arithmetic and measurement is invaluable. The pupils saw this in measuring distances, comparing weights, and making observations of the data.

In the search for a general pattern or for a new combination of facts, the medium of inductive reasoning is used. Frequently the reasoning process is described in mathematical language which is particularly appropriate for describing general characteristics and properties.

Ordinarily, the general rule itself is described best by a mathematical formula. In interpreting the general pattern which has been described in mathematical language, graphing is often effective. We have seen in our discussions of the lever that the use of a graph gives a convenient picture of a great deal of information.

At many places in the discussion of the lever, use was made of deductive reasoning. Sometimes, in fact, it is possible to develop a general law in mathematical terms entirely by using deductive reasoning and mathematical operations based on certain physical assumptions.

Throughout the entire fibre of our scientific activity and general civilization, mathematics plays a dominant role in analysis, description, interpretation, and generalization.

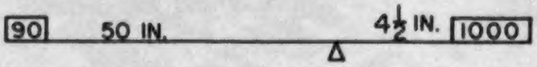
The historical examples of the text all involve simple mathematics which the pupils should be able to appreciate. You may wish to emphasize the increasingly complex nature of present-day scientific discoveries, many of which use new kinds of advanced mathematics. Class study of the current news of science might reveal a number of good illustrations.

The value of the chapter lies in the understanding which it provides of the way mathematics works with science. There is no particular value in requiring pupils to solve many numerical examples using the principle of the lever unless they need practice in computation. It is difficult to test, in the usual way, the ideas the chapter is designed to present; but a few sample questions for the chapter are included.

Answers to Exercises 14-6

1. Some seesaws are made so that they can be shifted on the fulcrum in order to alter the distances of the riders from the fulcrum and still permit them to sit at the ends of the seesaw. In this way, riders of different weights can balance each other without moving from the ends of the seesaw.

2. 9 yd.
3. 75 pounds
4. $7\frac{1}{2}$ ounces
5. 2 d
6. With a lever on a fulcrum at the proper place, a 90 pound girl can lift 1,000 pounds. For example:



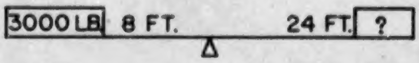
7. About 2 ft. from the fulcrum (1.8 ft.)
8. 37.5 pounds
9. 1100 pounds
10. (a) and (b)

Sample Test Questions
(Answers are at End)

1. **If two objects at the opposite ends of a seesaw balance, and the weight of one is halved, what must be done to the other to preserve the balance?**
2. If two boys on the opposite sides of a seesaw balance, and one moves twice as far as he was from the fulcrum, while the other does not move, what happens to the seesaw?

Find the missing value in each case:

3.



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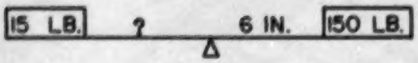
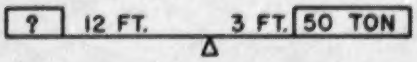
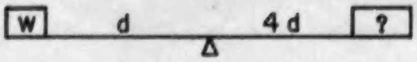

15 LB.
?
6 IN.
150 LB.

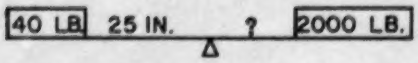
5.


?
12 FT.
3 FT.
50 TON

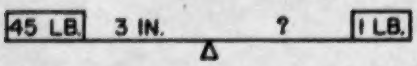
6.



7.


40 LB.
25 IN.
?
2000 LB.

8.


45 LB.
3 IN.
?
1 LB.

9. Tell how to balance a 60 pound boy and his 40 pound sister if she sits 6 ft. from the center support of the seesaw.

10. Complete the table

w =	9	9	9	9	9	9
d =	8	8	8	8	8	8
W =	2		18		24	
D =		6		16		$\frac{1}{12}$

11. If WD = 30, what happens to D as W increases? What happens to W as D increases?

12. If WD = 75 what is D when W is 15? What is W when D is 10?

13. Which of the points below are on the graph of WD = 63?

(a) (9, 7) (b) (10, 6) (c) (8, 8)

14. State one or two other formulas (similar to WD = 100) in which one value decreases as the other increases.

Answers to Sample Questions

1. Halve the weight of the other or double the corresponding distance from the fulcrum.

2. It is out of balance and the boy who did not move is higher than the other.

3. 1000 lb.

4. 60 inches

5. $12\frac{1}{2}$ tons

6. $\frac{W}{4}$ or $\frac{1}{4}$ W

7. $\frac{1}{2}$ in.

8. 135 in.

9. Seat the boy 4 ft. from the fulcrum.

10.

W		12		$4\frac{1}{2}$		864
D	36		4		3	

11. D decreases. W decreases.

12. W is 5. D is $7\frac{1}{2}$.

13. (a)

14. rt = 100 lw = 100 rp = 100
(where any constant may be used where 100 appears.)